1. *Fundamental Differential Equations in the Theory of Conformal Mapping.*

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(Comm. by T. KUBOTA, M.J.A., Jan. 12, 1949.)

1. Let $\mathfrak{S}$ be the family of analytic functions $F(z)$ which are regular and schlicht in the interior of the unit circle E: $|z|<1$ and further are normalized at the origin in such a way that $F(0)=0$, $F'(0)=1$. The theory of this family has been developed by various methods. Among them, one based upon the so-called Löwner's differential equation[1] on bounded slit mapping of E has been, especially first by Sovietic mathematicians G. M. Golusin, J. Basilewitsch etc., shown to be very fruitful. Let B be a bounded slit domain obtained from $|w|<1$ by cutting it along a Jordan arc which lies in $|w|<1$ save an end-point and does not pass through the origin. The mapping function $w=f(z)$, $f(0)=0$, $f'(0)=e^{-t_0}$, of E onto B is then regarded as the integral $f(z)=f(z, t_0)$ of the so-called Löwner's differential equation.

$$\frac{\partial f(z,t)}{\partial t}=-f(z,t)\frac{1+\kappa(t)f(z,t)}{1-\kappa(t)f(z,t)} \qquad (0\leqq t\leqq t_0) \tag{1.1}$$

with initial condition $f(z, 0)=z$, $\kappa(t)$ being a continuons function weose absolute value is identically equal to unity. Each function $w_t=f(z, t)$, for which $f(0, t)=0$ and $f'(0, t)=e^{-t}$, gives also a bounded slit mapping of E. Introduce now a new family of slit mapping functions $\{h(z, t)\}$ $(0\leqq t\leqq t_0)$ by functional relation

$$f(z)=h(f(z,t),t). \tag{1.2}$$

Then the differential equation for this family becomes

$$\frac{\partial h(z,t)}{\partial t}=z\frac{1+\kappa(t)z}{1-\kappa(t)z}\frac{\partial h(z,t)}{\partial z} \qquad (t_0\geqq t\geqq 0) \tag{1.3}$$

with boundary conditions $h(z, t_0)=z$ and $h(z, 0)=f(z)$.

Now, remembering the structure of Löwner's differeutial equation, we may expect that analogous equations can be constructed in various ways from more general point of view. We consider, in general, a function $w=F(z)$ which maps E onto a given simply connected domain D in the w-plane. Suppose that a family of simply connected domains $\{D_t\}$ with a real parameter t $(0\leqq t\leqq t_0)$ be constructed in such a way that D_0 and D_{t_0} coïncide with the domains $|w|<1$ and D

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1) K. Löwner, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I. Math. Ann. **89** (1923), 103–121.

respectively. Let $w=F(z, t)$ be a function mapping E onto D_t. If, then, $F(z, t)$ is differentiable with respect to the parameter t, a differential equation, e. g., of the form

$$\frac{\partial F(z, t)}{\partial t}=\Phi(t, z, F(z, t), F'(z, t)) \qquad (0\leqq t\leqq t_0)$$

may be expected, and its integral $F(z, t_0)$ with initial condition $F(z, 0)=z$ will give a required mapping function $F(z)$. And the construction of such a family of domains will be possible with a large freedom of selection. The Löwner's equation is also regarded to belong to this category. From such a point of view, Schaeffer and Spencer[2] have recently obtained a differential equation of this kind, and shown its utility for coefficient problems of the family $\mathfrak{S}$.

Now, the Löwner's equation can be briefly derived by means of Poisson integral formula for analytic functions regular in E[3]. This method is also shown to be useful to obtain a Julia-Biernacki's formula for variation of mapping function caused by a variation of image-domain.[4] In the present Note we shall show that the Schaeffer-Spencer's equation can briefly be derived by means of the last-mentioned formula, and further will obtain a corrssponding result for conformal mapping of doubly-connected domains.

2. If $F(z)$ belongs to $\mathfrak{S}$, so also does the function $\alpha^{-1}F(\alpha z)$ for any real positive constant α less than unity, and the relation

$$\lim_{\alpha\to 1-0}\alpha^{-1}F(\alpha z)=F(z)$$

holds in E uniformly in the wider sense. The image-domain of E by each mapping $w=\alpha^{-1}F(\alpha z)$ with $0<\alpha<1$ always contains $w=0$ and is bounded by a regular analytic and closed Jordan curve. Let $\mathfrak{A}$ be the sub-family of $\mathfrak{S}$ consisting of all functions, each of which possesses such a smooth curve as the boundary of its image-domain. The original family $\mathfrak{S}$ is a normal family, in which the family $\mathfrak{A}$ is everywhere dense. Hence, when the extremal problems in $\mathfrak{S}$ for continuous functionals, such as distorsion or coefficient problems, are concerned, we can restrict ourselves to its proper sub-family $\mathfrak{A}$ each member of which is regular on the closed circular disc E': $|z|\leqq 1$ due to its analytic continuability. The Schaeffer-Spencer's results relates to $\mathfrak{A}$, which states as follows:

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2) A. C. Schaeffer and D. C. Spencer, The coefficients of schlicht functions, II. Duke Math. Journ. **12** (1945), 107–125.

3) Y. Komatu, Über einen Satz von Herrn Löwner. Proc. Imp. Acad. Tokyo **16** (1940), 512–514.

4) See Y. Komatu, Sur la variation d'une fonction de représentation conforme, lorsque le domaine varie. Proc. Imp. Acad. Tokyo **19** (1943), 599–608 in which the detailed references are contained.

THEOREM 1. *Let $F(z)$ be a function in the family $\mathfrak{A}$, and let e^{t_0} be any positive number greater than the maximum value of $|F(z)|$ in E'. Then there is a family $h(z, t)$ defined for $z \in E$ and $0 \leqq t \leqq t_0$ which satisfies the differential equation*

$$\frac{\partial h(z, t)}{\partial t} = zp(z, t)\frac{\partial h(z, t)}{\partial z}$$

with boundary conditions $h(z, t_0)=z$ and $h(z, 0)=f(z)\equiv e^{-t_0}F(z)$. Here $p(z, t)$ is for each t, an analytic function of z, regular and with positive real part in E and further equal to 1 at $z=0$.

Proof. We notice first of all, according to a theorem of Schwarz, that $1 = F'(0) < e^{t_0}$ or $t_0 > 0$. Let D be the image-domain of E by the mapping $w=f(z)$. Then its boundary C is a regular analytic and closed Jordan curve, contained in $|w|<1$ and enclosing the origin. Let the *modulus* of (doubly-connected) ring domain between C and $|w|=1$ be $\lg(1/r_0)(>0)$ and denote by $w=R(z^*)$ a function which maps $r_0<|z^*|<1$ onto this ring domain in such a manner that $|z^*|=1$ and $|z^*|=r_0$ correspond to $|w|=1$ and C respectively. It is well-known that such a mapping is possible and is uniquely determined except a rotation in the z^*-plane about the origin. Let C_r be the image of $|z^*|=r$ ($r_0 \leqq r \leqq 1$) by this mapping; in particular, C_{r_0} and C_1 represent just C and $|w|=1$ respectively. We denote by

$$w=g(z, r) \qquad \left(g(0, r)=0,\ g'(0, r)\equiv\frac{\partial g}{\partial z}(0, r)>0\right)$$

the mapping function of E onto the (bounded) domain Δ_r with boundary C_r; then we have, in particular,

(2.2) $$g(z, r_0)=f(z), \qquad g(z, 1)=z.$$

Let now $\omega=g(\xi, r)(|\xi|=1)$ be any point on C_r, and put

$$\xi^*=R^{-1}(\omega), \qquad |\xi^*|=r.$$

All functions in question behave regularly even on the boundaries of their respective domains. Hence, the perpendicular displacement from C_r to $C_{r+\delta r}$ is given by

$$\delta\omega=R'(\xi^*)\delta\xi=R'(\xi^*)\frac{\xi^*}{r}\delta r,$$

an infinitesimal quantity of order higher than that of δr being neglected. Julia-Biernacki's formula then gives

(2.3) $$\begin{aligned}\delta g(z, r)&\equiv g(z, r+\delta r)-g(z, r)\\ &=\frac{zg'(z, r)}{2\pi}\int_{C_r}\frac{\xi+z}{\xi-z}\,\frac{\delta\omega d\omega}{\xi^2 g'(\xi, r)^2}(1+o(1)),\end{aligned}$$

the integral being taken round C_r, as usual, in the positive sense and $o(1)$ denoting a quantity which tends to zero with δr in E uniformly in the wider sense. On the other hand, as we have

$$d\omega = g'(\xi, r)d\xi = g'(\xi, r)i\xi d\theta \qquad (\xi = e^{i\theta})$$

and for $\delta r > 0$

$$\delta\omega = g'(\xi, r)\delta\xi = g'(\xi, r)\xi\,|\,\delta\xi\,|,$$

the quantity

$$\frac{1}{i}\,\frac{\delta\omega}{\xi^2 g'(\xi, r)^2}\,\frac{d\omega}{d\theta} = \frac{\xi^* R'(\xi^*)\delta r}{r\xi g'(\xi, r)} = |\,\delta\xi\,|$$

is always real-positive, and expresses the magnitude of perpendicular displacement at ξ. This fact is also seen by remebering that Δ_r is contained, its boundary inclusive in the interior of $\Delta_{r+\delta r}$ for $\delta r > 0$ and further $\delta\omega$ denotes an outward displacement. Let $u(z, r)$ be a function regular in E whose real part possesses the quantity $\xi^* R'(\xi^*)/(r\xi g'(\xi, r))$ as its boundary values. Then $\Re u(z, r)$ is positive throughout E and the function itself is expressed by Poisson formula:

$$u(z, r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\xi + z}{\xi - z}\,\frac{\xi^* R'(\xi^*)}{r\xi g'(\xi, r)}\,d\theta \qquad (\xi = e^{i\theta})$$

$$= \frac{1}{2\pi i}\int_{C_r} \frac{\xi + z}{\xi - z}\,\frac{\delta\omega d\omega}{\delta r \xi^2 g'(\xi, r)^2}.$$

Comparing this with (2.3), we have

$$\frac{\delta g(z, r)}{\delta r} = zg'(z, r)u(z, r)(1 + o(1))$$

and, making $\delta r \to 0$,

$$(2.4) \qquad \frac{\partial g(z, r)}{\partial r} = zu(z, r)\frac{\partial g(z, r)}{\partial z}.$$

Divide the last equation by z and put $z = 0$. Then we have

$$(2.5) \qquad \frac{\partial g'(0, r)}{\partial r} = u(0, r)g'(0, r);$$

the relation which can also be derived strictly by a method similar to the above mentioned one which has given (2.4). Now since, by (2.2), $g'(0, r_0) = f'(0) = e^{-t_0}$ and $g'(0, 1) = 1$, we get from (2.5)

$$g'(0, r) = \exp\left(-t_0 + \int_{r_0}^{r} u(0, r)\,dr\right) = \exp\int_1^r u(0, r)dr.$$

But $u(0, r)$ is always real-positive, and so $g'(0, r)$ is a continuously differentiable function which varies, as r varies from r_0 to 1, monotone in the strict sense from e^{-t_0} to 1; this monotony is also an immediate consequence from a Schwarz-Lindelöf's theorem. Hence, the relation

$$e^{t - t_0} = g'(0, r), \qquad t = t_0 + \lg g'(0, r)$$

defines a continuously differentiable function $t=t(r)$ for $r_0 \leqq r \leqq 1$ which increases strictly monotone with r. It is obvious that $t(r_0)=0$ and $t(1)=t_0$. Now, with this substitution of parameter, we put

$$(2.6) \qquad h(z,t)=g(z,r(t)), \qquad p(z,t)=u(z,r(t))\frac{dr(t)}{dt},$$

then, we get the reqired equation (2.1). Here $p(z,t)$ is regular in E and further, since $\Re u(z,r)$ is positive in E and $dr/dt>0$, it is of positive real part in E. On the other hand, the relation $u(0,r)=d \lg g'(0,r)/dr=dt/dr$ implies that

$$p(0,t)=u(0,r(t))\frac{dr(t)}{dt}=1.$$

Finally, from (2.6) we have

$$h(z,t_0)=g(z,r(t_0))=g(z,1)=z,$$
$$h(z,0)=g(z,r(0))=g(z,r_0)=f(z),$$

and the proof of the theorem is completed.

The equation (2.1) appearing in the theorem just proved corresponds to Löwner's equation (1.3). In fact, the latter is obtained from (2.1) by putting $(1+\kappa(t)z)/(1-\kappa(t)z)$ in place of $p(z,t)$. This particular function is regular, of positive real part in E and equal to 1 at $z=0$; the only singular character is that it possesses a pole $\kappa(t)$ on $|z|=1$.

Löwner[5] derived his coefficient theorem $|F'''(0)|/3!\leqq 3$ by making use of (1.3). In this connection Schaeffer and Spencer[6] have noticed that the same result can also be derived by means of (2.1). Though they then have emphasized the utility of the equation (2.1), the function $(1+\kappa z)/(1-\kappa z)$ possesses all characteristic properties of $p(z,t)$ except only a singularity on $|z|=1$ and consequently the range of applicability of (2.1) will not be essentially wider than that of (1.3).

3. For conflicient problems, the equation (2.1) is often applied effectively. But, as we have intention to discuss distorsion problems of $\mathfrak{S}$ in a later paper, we will derive here an analogous differential equation corresponding to (1.1).

THEOREM 2. *Suppose that all the assumptions of the preceding theorem are satisfied. Then the function* $f(z)\equiv e^{-t_0}F(z)$ *is determined as the integral* $f(z)=f(z,t_0)$ *of the differential equation*

$$(3.1) \qquad \frac{\partial f(z,t)}{\partial t}=-f(z,t)k(z,t) \qquad (0\leqq t\leqq t_0)$$

5) K. Löwner, loc. cit.
6) A. C. Schaeffer and D. C. Spencer, loc. cit.

with initial condition $f(z, 0)=z$. *Here* $k(z, t)$ *is a function possessing the properties as assigned above for* $p(z, t)$; *more precisely* $k(z, t)\equiv p(f(z, t), t)$.

Proof. Retaining the notations in the proof of the preceding theorem, as $w=h(z, t)$ maps E onto $\Delta_{r(t)}$, we may write here symbolically $\Delta_{r(t)}=h(E, t)$. Let D_t be the domain which corresponds to $D(\subset\Delta_{r(t)})$ by this mapping, and hence

$$D=h(D_t, t).$$

Considering now D_t to be laid on the w-plane, let the function mapping E onto D_t be

$$w=f(z, t) \qquad \left(f(0, t)=0, f'(0, t)\equiv\frac{\partial f}{\partial z}(0, t)>0\right).$$

Then $D_t=f(E, t)$, and so

$$D=h(f(E, t), t).$$

On the other hand, since $D=f(E)$, we get, by the uniqueness theorem of mapping, the identical relation (1.2), i. e.

$$(3.2) \qquad f(z)=h(f(z, t), t).$$

In particular, for $t=0$ and $t=t_0$, $h(w, t)$ reduces to $f(w)$ and to w respectively, and hence

$$f(z)=h(f(z, 0), 0)=f(f(z, 0))$$
$$=h(f(z, t_0), t_0)=f(z, t_0).$$

Therefore, we get

$$f(z, 0)=z, \qquad f(z, t_0)=f(z).$$

Now, by (3.2), for the function $w=f(z, t)$ it holds good the relation

$$0=\frac{\partial h(w, t)}{\partial w}\frac{\partial w}{\partial t}+\frac{\partial h(w, t)}{\partial t}.$$

On the other hand, in the equation (2.1), writing w in place of z, we get

$$\frac{\partial h(w, t)}{\partial t}=wp(w, t)\frac{\partial h(w, t)}{\partial w}.$$

The function $h(w, t)$ being schlicht, $\partial h/\partial w$ vanishes nowhere in $|w|<1$, and hence by eliminating $\partial h/\partial t$ from both equations above, we have finally the differential equation

$$\frac{\partial w}{\partial t}=-wp(w, t) \qquad (w=f(z, t)).$$

Since $|f(z, t)|<1$, the function

$$(3.3) \qquad k(z, t)=p(f(z, t), t)$$

possesses the properties just mentioned above. This proves the theorem.

4. In a previous paper,[7] the present author has shown that the Julia-Bier-

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7) Y. Komatu, loc. cit. [4]

nacki's formula can be generalized to the case of doubly-connected domains. Corresponding to it, we shall generalize, in the following lines, the above theorems also to the case of doubly-connected domains The results which will be obtained correspond to those which the author has given as a generalization of Löwner's equations.[8]

We take a concentric annular ring as standard (doubly-connected) ring domain. Let, on the w-plane, a ring domain D be given whose boundary is composed of the circumference $|w|=1$ and a simple closed curve C separating it from the point at infinity. Suppose, for the sake of brevity, that C is a regular analytic curve. By normal family property this restriction is not essential for extremal problems concerning continuous functionals. Let the modulus of D be $\lg Q^{-1}(0<Q<1)$, and let a function

$$(4.1) \qquad w=f(z) \qquad (f(1)=1)$$

map $1<|z|<Q^{-1}$ schlicht onto D. If we take an annular ring $1<|w|<Q_0^{-1}$ containing $D+C$, then the monotony of moduli implies $Q_0<Q$. Let the modulus of the ring domain between C and $|w|=Q_0^{-1}$ be $\lg P_0^{-1}(0<P_0<1)$ and the function mapping $1<|z^*|<P_0^{-1}$ onto this ring domain be

$$w=R(z^*) \qquad (R(P_0^{-1})=Q_0^{-1}).$$

For each p $(R_0\leqq p\leqq 1)$, let Γ_p be the image of $|z^*|=p^{-1}$ by this mapping, and the modulus of the ring domain Δ_p between $|w|=1$ and Γ_p be denoted by $\lg q^{-1}$. By the monotony character of moduli it holds good $Q_0\leqq q\leqq Q$. Denote by

$$w=h(z, q) \qquad (h(1, q)=1)$$

the function which maps $1<|z|<q^{-1}$ onto the last mentioned ring domain; then we have in particular

$$(4.2) \qquad h(z, Q_0)=z, \qquad h(z, Q)=f(z).$$

Our present object is to obtain a differential equation for the family $h(z, q)$ regarded as the function of the parameter q.

THEOREM 3. *The family $h(z, q)$ defined for $1<|z|<q^{-1}$ and $Q_0\leqq q\leqq Q$ as above satisfies the differential equation*

$$(4.3) \qquad \frac{\partial h(z, q)}{\partial \lg q}=-zL(z, q)\frac{\partial h(z, q)}{\partial z}$$

with boundary conditions (4.2) *Here $L(z, q)$ is, for each q, a regular analytic function of z in $1<|z|<q^{-1}$ satisfying the inequalities*

$$(4.4) \qquad 2\Re A(-r, q^{-1}; q)\leqq \Re L(z, q)\leqq 2\Re A(r, q^{-1}; q) \qquad (|z|=r),$$

where the function $A(z, \xi; q)$ is given by

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8) Y. Komatu, Untersuchungen über konforme Abbildung von zweifach zusammenhängenden Gebieten. Proc. Phys.-Math. Soc. Japan **25** (1934), 1–42.

$$(4.5)\qquad A(z,\xi;\,q)\equiv\frac{1}{i}\left(\zeta\left(i\lg\frac{z}{\xi};q\right)-\zeta\left(i\lg\frac{1}{\xi};q\right)\right)-\frac{\eta_1}{\pi}\lg z,$$

ζ-function being that of Weierstrassian theory of elliptic functions with primitive periods 2π and $2i\lg q^{-1}$; η_1 also depending on q.

Proof. All functions in question are regular on the boundaries of their respective domains. The quantity

$$p=p(q)\equiv\left|R^{-1}\left(h\left(\frac{1}{q},q\right)\right)\right|$$

increases strictly monotone with q. We put $\Gamma_{p(q)}=C_q$. Let ξ^* be any point on $|z^*|=p^{-1}$ and put

$$R(\xi^*)=\omega=h(\xi,q).$$

Then $|\xi^*|=p^{-1}$, $\omega\in C_q$ and $|\xi|=q^{-1}$. Denote an inward normal displacement at ξ^* by

$$\delta\xi^*=\frac{\xi^*}{p^{-1}}\delta p^{-1}=-\xi^*p^{-1}\delta p;$$

then, the mapping being conformal, there corresponds an inward normal displacement on C_q given by

$$\delta\omega=R'(\xi^*)\delta\xi^*=-R'(\xi^*)\xi^*p^{-1}\delta p.$$

On the other hand, let $\delta\xi$ denote the corresponding displacement on the z-plane; though it is also an inward normal displacement on $|z|=q^{-1}$ at ξ, the point $\xi+\delta\xi$ does not necessarily lie on $|z|=(q+\delta q)^{-1}$. But since

$$\delta\omega=h'(\xi,q)\delta\xi=-h'(\xi,q)q\xi|\delta\xi|,$$

the variational equation, derived in a previous paper,[9] gives

$$\delta h(z,q)\equiv\frac{\partial h(z,q)}{\partial q}\delta q$$

$$=-zh'(z,q)\left\{\frac{1}{\pi}\int_{C_q}\left(\zeta\left(i\lg\frac{z}{\xi};q\right)-\zeta\left(i\lg\frac{1}{\xi};q\right)\right)\frac{\delta\omega d\omega}{\xi^2h'(\xi,q)^2}-\frac{2\eta_1\lg z}{\pi}\frac{\delta q}{q}\right\},$$

here an infinitesimal of higher order is, of course, omitted and the integral is taken round C_q in the positive sense. But, as has been shown in the above cited paper, we have

$$-\frac{\delta q}{q}=\frac{1}{2\pi i}\int_{C_q}\frac{\delta\omega d\omega}{\xi^2h'(\xi,q)^2}$$

and hence

9) Loc. cit. 4)

$$\delta h(z;q)=\frac{zh'(z,q)}{\pi i}\int_{C_q}\Big(\frac{1}{i}(\zeta(i\lg\frac{z}{\xi};q)-\zeta(i\lg\frac{1}{\xi};q))$$
$$-\frac{\eta_1}{\pi}\lg z\Big)\frac{\delta\omega d\omega}{\xi^2h'(\xi,q)^2}.$$

Now, putting $\xi=q^{-1}e^{i\theta}$, we obtain

$$\frac{\delta\omega d\omega}{i\xi^2h'(\xi,q)^2}=\frac{-h'(\xi,q)q\xi\,|\,\delta\xi\,|\cdot h'(\xi,q)i\xi d\theta}{i\xi^2h'(\xi,q)^2}=-q\,|\,\delta\xi\,|\,d\theta,$$

a quantity which is real-negative. As $\delta q\to 0$, we get

$$\frac{|\,\delta\xi\,|}{\delta\lg q}=-\frac{h'(\xi,q)q\xi}{\delta\omega}\Big/\frac{\delta q}{q}=\frac{R'(\xi^*)\xi^*p^{-1}\delta p}{h'(\xi,q)\xi\delta q}\to\frac{R'(\xi^*)\xi^*p^{-1}p'(q)}{h'(\xi,q)\xi}$$

and finally, by using the definition (4.5),

$$\frac{\partial h(z,q)}{\partial\lg q}=-\frac{zh'(z,q)}{\pi}\int_0^{2\pi}A(z,\xi;q)\frac{qR'(\xi^*)\xi^*p^{-1}p'(q)}{h'(\xi,q)\xi}d\theta.$$

It is easily shown that $A(z,\xi;q)$ is expanded in infinite series of the form

$$A(z,\xi;q)=\sum_{n=1}^{\infty}\frac{z^n-1}{1-q^{2n}}\Big(\frac{1}{\xi^n}+\frac{q^{2n}\xi^n}{z^n}\Big),$$

and hence, putting $z=re^{i\varphi}(1<r<q^{-1})$ and $\xi=q^{-1}e^{i\theta}$, we get

$$\Re A(re^{i\varphi},q^{-1}e^{i\theta},q)=\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\Big(r^n-\frac{1}{r^n}\Big)\cos n(\theta-\varphi).$$

Since $q<rq<1$, as shown previously,[10] the last quantity is an even function of $\theta-\varphi$ and decreases, as $\theta-\varphi$ varies from 0 to π, monotone in the strict sense; thus we have in particular

$$\Re A(-r,q^{-1};q)\leqq\Re A(re^{i\varphi},q^{-1}e^{i\theta};q)\leqq\Re A(r,q^{-1};q).$$

On the other hand, since

$$-\delta\log q=\frac{1}{2\pi i}\int_{C_q}\frac{\delta\omega d\omega}{\xi^2h'(\xi,q)^2}=-\frac{q}{2\pi}\int_0^{2\pi}|\,\delta\xi\,|\,d\theta,$$

we have a relation

$$\int_0^{2\pi}\frac{qR'(\xi^*)\xi^*p^{-1}p'(q)}{h'(\xi,q)\xi}d\theta=2\pi.$$

Hence, if we define the function $L(z,q)$ by the equation (4.3), the inequalities (4.4) hold good for this function. This proves the theorem.

5. The above mentioned theorem 2 can also be generalized in quite similar manner. The result may be stated as follows:

THEOREM 4. *Under the same assumptions as in the previous theorem, the function $f(z)$ is determined as the integral $f(z)=f(z,Q_0)$ of the differen-*

10) Loc. cit. [8]

tial equation

$$\frac{\partial \lg f(z, q)}{\partial \lg q} = K(z, q) \qquad (Q \geqq q \geqq Q_0) \tag{5.1}$$

with initial condition $f(z, Q) = z$, $K(z, q)$ *being defined by*

$$K(z, q) = L(f(z, q), q). \tag{5.2}$$

Proof. The argument is quite similar as that of the proof of theorem 2. We have, in fact, only to consider the family $f(z, q)$ defined by the relation

$$f(z) = h(f(z, q), q),$$

for which the relations (5.1) and (5.2) are almost immediate consequences of (4.3).