18. *Fundamental Theory of Toothed Gearing (III).*

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Suppose that a pair of pitch curves K_1 and K_2 and a pair of profile curves F_1 and F_2 invariably connected with K_1 and K_2 be given. Let P be a common pitch point at a certain instant, C be the point of contact of F_1 and F_2 corresponding to P. Suppose that after infinitesimal time interval dt two points P_1 and P_2 on K_1 and K_2 and two points C_1 and C_2 on F_1 and F_2 may respectively come to the point of contact. Denote by ds the length of the arc PP_1, and consequently that of PP_2, and by dp_1 and dp_2 the lengths of the arcs CC_1 and CC_2 respectively. The pitch curve K is oriented and accordingly ds has a sign positive or negative. We shall give also a sign to dp; dp is positive or negative according as the part of arc dp of the profile curve F is of positive or negative type.

§ 1. Sliding of profile curves.

At the sliding contact motion of F_1 and F_2 during the time dt the point C on F_1 slides along F_2 for the distance $dp_2 - dp_1$, and consequently its velocity v_{p1} is given by

$$(1)_1 \qquad v_{p1} = \frac{dp_2 - dp_1}{dt}.$$

v_{p1} is named the velocity of sliding of F_1 (at the point C on F_2). In like manner the velocity of sliding of F_2 may be defined:

$$(1)_2 \qquad v_{p2} = \frac{dp_1 - dp_2}{dt}.$$

Evidently v_{p1} and v_{p2} have the same absolute value and the different signs.

Denoted by ω_1 and ω_2 respectively the instant angular velocities of K_1 and K_2 at their rolling contact motions, and we say that the sign of the angular velocity ω is positive or negative according as K rotates clockwise or counter-clockwise. Denoting by a_1 and a_2 the radii of curvature of K_1 and K_2 respectively at the instant common pitch point P we have

$$(2) \qquad \omega_1 = \frac{1}{a_1}\frac{ds}{dt}, \qquad \omega_2 = \frac{1}{a_2}\frac{ds}{dt}.$$

Let ω denote the relative rolling angular velocity of K_1 to K_2, then obviously $\omega = \omega_1 - \omega_2$ and accordingly from (2)

$$\omega=\left(\frac{1}{a_1}-\frac{1}{a_2}\right)\frac{ds}{dt}\,. \tag{3}$$

Next, let r be the length of the segment of the straight line connecting P with the point of contact C of F_1 and F_2, and give r a positive or negative sign in such a manner as we have explained at the beginning of the report (II), then the velocity v_{p1} of C is represented by $r\omega$, that is,

$$v_{p1}=\left(\frac{1}{a_1}-\frac{1}{a_2}\right)r\,\frac{ds}{dt}\,. \tag{4$_1$}$$

Similarly

$$v_{p2}=\left(\frac{1}{a_2}-\frac{1}{a_1}\right)r\,\frac{ds}{dt}\,. \tag{4$_2$}$$

From (4) follows immediately:

Profile curves make rolling contact motion without sliding if and only if they coincide with their pitch curves.

Now, we may consider the acceleration of C: w_p. Denoting by w_t and w_n its tangential and normal component, we have, as is well known,

$$w_t=\frac{dv_p}{dt}\,,\qquad w_n=\frac{v_p^2}{m}\,, \tag{5}$$

where m denotes the radius of curvature of the mate of F at C.

In particular, when K_1 and K_2 are both circles and their rotations are of constant velocities, we have by (4) and (5) above obtained, and (2) in the report (II)

$$(6)_1\qquad \begin{aligned} w_{t1}&=-sgn(\theta)\left(\frac{1}{a_1}-\frac{1}{a_2}\right)\left(\frac{ds}{dt}\right)^2\cos\theta\,,\\ w_{n1}&=\left(\frac{1}{a_1}-\frac{1}{a_2}\right)^2\left(\frac{ds}{dt}\right)^2\frac{r^2}{m_2}\,,\end{aligned}$$

and

$$(6)_2\qquad \begin{aligned} w_{t2}&=-sgn(\theta)\left(\frac{1}{a_2}-\frac{1}{a_1}\right)\left(\frac{ds}{dt}\right)^2\cos\theta\,,\\ w_{n2}&=\left(\frac{1}{a_2}-\frac{1}{a_1}\right)^2\left(\frac{ds}{dt}\right)^2\frac{r^2}{m_1}\,,\end{aligned}$$

and further from $(6)_1$ and $(6)_2$

$$\frac{w_{n1}}{w_{n2}}=\frac{m_1}{m_2}\,. \tag{7}$$

Therefore, from (4), (6) and (7) is derived the following:

Theorem 1. *Given a pair of pitch circles which make rolling contact motion with constant velocity of rotation and a pair of profile curves invariably connected with those pitch circles. The velocities of sliding at any point of contact of the profile curves are proportional to the distance from the point to the pitch point corresponding to it, and the tangential components of the accelerations are proportional to the cosine of the angle between the straight line connecting the point of contact with the pitch point corresponding to it and the common tangent to the pitch curves at the pitch point. Furthermore the ratio of the normal components of the accelerations is equal to the ratio of the radii of curvature of the profile curves.*

§ 2. The types of profile curves.

When we particularly adopt the rolling curve K_r as one of the pitch curves K_1 and K_2, we have from (4)

$$(8) \qquad \frac{dp_1}{dt}=\left(\frac{1}{a_r}-\frac{1}{a_1}\right)r\frac{ds}{dt}, \qquad \frac{dp_2}{dt}=\left(\frac{1}{a_r}-\frac{1}{a_2}\right)r\frac{ds}{dt}.$$

Consequently we have the following relations (9) between the arc length ds of the pitch curve K and the arc length dp of the profile curve F corresponding to it:

$$(9) \qquad dp_1=\left(\frac{1}{a_r}-\frac{1}{a_1}\right)rds, \qquad dp_2=\left(\frac{1}{a_r}-\frac{1}{a_2}\right)rds.$$

In accordance with (9) we can derive the following theorem concerning the types of roulettes, namely, of profile curves.

Theorem 2. *Let a curve K_r with the natural equation $a_r=a_r(s)$ roll without sliding along a curve K with the natural equation $a=a(s)$. In the range of s, where the curvature $\frac{1}{a_r(s)}$ of K_r is larger than K's; $\frac{1}{a_r(s)}>\frac{1}{a(s)}$, the roulette F drawn by a point C fixed at K_r is of positive type as far as the point C exists on the left side of the common tangent of K_r and K at the common pitch point, and negative type as far as C exists on the right side. In the range, where $\frac{1}{a_r(s)}<\frac{1}{a(s)}$, the converse holds.*

Moreover, we have the following theorem concerning the assortion of the types of a pair of profile curves.

Theorem 3. *Let the natural equations of a pair of profile curves K_1 and K_2 and rolling curve K_r be $a_1=a_1(s)$, $a_2=a_2(s)$ and $a_r=a_r(s)$ respectively. In the range of s, where the curvature $\frac{1}{a_r(s)}$ of K_r is larger or smaller than both of the radii of curvature $\frac{1}{a_1(s)}$ and $\frac{1}{a_2(s)}$ of K_1 and K_2, in other* *words, both K_1 and K_2 exist on one side of K_r, the same type parts of F_1 and F_2 are in mesh, and in the range, where $\frac{1}{a_r(s)}$ exists between $\frac{1}{a_1(s)}$ and $\frac{1}{a_2(s)}$, that is, K_r exists between K_1 and K_2, the different type parts of F_1 and F_2 are in mesh.*

By Theorem 5 in the report (II), a_r, the radius of curvature K_r, is equal to the length of the segment cutten off by the normal to the path of contact Γ on the perpendicular P_0N_0 to the initial line P_0T_0 at the pole P_0. Consequently, when both the pitch curves K_1 and K_2 are circles, we can state Theorem 3 in the following manner.

Theorem 4. *Given a pair of pitch circles O_1 and O_2 touching at a point P_0 and a path of contact Γ settled at their connnon tangent P_0T_0. Let M be the point at which a normal to Γ intersects the straight line O_1O_2 connecting the centers of circles O_1 ond O_2. As far as M exists on the one of the two parts of the center line O_1O_2 divided by the two points O_1 and O_2, on which part the point P_0 is contained, are in mesh the same type parts of the profile curves corresponding to Γ. When M exists on the part not containing P_0, are in mesh the different type parts of the profile curves are in mesh.*

§ 3. Specific slidings of profile curves.

Now we proceed to define

$$\sigma_1=\frac{dp_2-dp_1}{dp_1}, \qquad \sigma_2=\frac{dp_1-dp_2}{dp_2} \tag{10}$$

and name them respectively the specific slidings of the profile curves F_1 and F_2 (at the point C on F_2 and F_1). When the same type parts of F_1 and F_2 are in mesh, then σ_1 and σ_2 have different signs. When the different type parts are in mesh, σ_1 and σ_2 have the same sign and are both negative. In addition, from (10) evidently follows the relation:

$$\frac{1}{\sigma_1}+\frac{1}{\sigma_2}=-1. \tag{11}$$

From (1) and (10) we have

$$\sigma_1=v_{p1}\ \frac{dp_1}{dt}, \qquad \sigma_2=v_{p2}\Big/\frac{dp_2}{dt}. \tag{12}$$

Substituting (4) and (8) into (12), we have

$$\sigma_1=\sigma_1(s)=\frac{\dfrac{1}{a_1(s)}-\dfrac{1}{a_2(s)}}{\dfrac{1}{a_r(s)}-\dfrac{1}{a_1(s)}}, \qquad \sigma_2=\sigma_2(s)=\frac{\dfrac{1}{a_2(s)}-\dfrac{1}{a_1(s)}}{\dfrac{1}{a_r(s)}-\dfrac{1}{a_2(s)}}, \tag{12}$$

From this follows immediately the fact:

For any profile curves with the same pitch curves and rolling curve the specific slidings at the points of contact corresponding to the same pitch point are all equal, wherever a drawing point is set at the rolling curve.

When the equation $r=f(s)$ of the profile curve F is given, we have, substituting Equation (8) in the report (II) into the above equation (13),

$$(14)\quad \begin{aligned} \sigma_1=\sigma_1(s)&=\left(\frac{1}{a_1(s)}-\frac{1}{a_2(s)}\right)\Bigg/\left(\frac{1-\{f'(s)\}^2-f(s)\cdot f''(s)}{f(s)\sqrt{1-\{f'(s)\}^2}}-\frac{1}{a_1(s)}\right),\\ \sigma_2=\sigma_2(s)&=\left(\frac{1}{a_2(s)}-\frac{1}{a_1(s)}\right)\Bigg/\left(\frac{1-\{f'(s)\}^2-f(s)\cdot f''(s)}{f(s)\sqrt{1-\{f'(s)\}^2}}-\frac{1}{a_2(s)}\right). \end{aligned}$$

When the equation $r=g(\theta)$ of the path of contact is given, we have, substituting Equation (14) in the report (II) into the above equation (13)

$$(15)\quad \begin{aligned} \sigma_1=\sigma_1(\theta)&=\frac{\dfrac{1}{a_1(s(\theta))}-\dfrac{1}{a_2(s(\theta))}}{\dfrac{\sin\theta}{g(\theta)}+\dfrac{\cos\theta}{|g(\theta)|'}-\dfrac{1}{a_1(s(\theta))}},\\ \sigma_2=\sigma_2(\theta)&=\frac{\dfrac{1}{a_2(s(\theta))}-\dfrac{1}{a_1(s(\theta))}}{\dfrac{\sin\theta}{g(\theta)}+\dfrac{\cos\theta}{|g(\theta)|'}-\dfrac{1}{a_2(s(\theta))}}, \end{aligned}$$

where
$$s(\theta)=-\int\frac{|g(\theta)|'}{\cos\theta}\,d\theta.$$

§ 4. The radii of curvature of profile curves.

We denote by m the radius of curvature of the profile curve F at the point C on F. As we have defined, the infinitesimal arc dp of F is oriented, according to this orientation we give m a positive or negative sign by the method we have already explained at the beginning of the report (II). Then we have

$$(16)\quad \frac{\pm m}{r\pm m}=\frac{dp}{ds|\sin\theta|},\quad \text{namely,}\quad \frac{dp}{ds}=\frac{\pm m}{r\pm m}|\sin\theta|,$$

where we take, from double signs $\pm$ before m, $+$ if F is of positive type and $-$ if F of negative type. It follows from (8) and (16)

$$(17)\quad \frac{1}{a_r}-\frac{1}{a}=\left(\frac{1}{r}-\frac{1}{r\pm m}\right)|\sin\theta|.$$

This is the formula of Savary concerning the radius curvature of a roulette drawn by a point fixed at a curve K_r when K_r rolls without sliding along a curve K.

From (17) we can derive the relation between the radii of curvature m_1 and m_2 of a pair of profile curves F_1 and F_2 at a point of contact:

$$(18)\qquad \frac{1}{a_1}-\frac{1}{a_2}=\left(\frac{1}{r\pm m_1}-\frac{1}{r\pm m_2}\right)|\sin\theta|,$$

where out of the double signs before m_1 and m_2 — in total four signs —, we assort the same two if F_1 and F_2 are of the same type, and the different two if F_1 and F_2 are of the different types.

Substituting (17) and (18) into (13) we have

$$(19)\qquad \sigma_1=\frac{\dfrac{1}{r\pm m_1}-\dfrac{1}{r\pm m_2}}{\dfrac{1}{r}-\dfrac{1}{r\pm m_1}},\qquad \sigma_2=\frac{\dfrac{1}{r\pm m_2}-\dfrac{1}{r\pm m_1}}{\dfrac{1}{r}-\dfrac{1}{r\pm m_2}}.$$

Now we shall consider a profile curve F and a parallel profile curve F^* with the distance a from F. Denote by K_r and K_r^* respectively the rolling curves for F and F^*, and let $a_r=a_r(s)$ and $a_r^*=a_r^*(s)$ be the natural equations of K_r and K_r^* respectively. By Equation (5) in the report (II) we can derive

$$(20)\qquad \frac{1}{a_r^*}-\frac{1}{a_r}=\left(\frac{1}{r+a}-\frac{1}{r}\right)|\sin\theta|.$$

Comparing (20) with (17), we obtain the following

Theorem 5. *Let F and F^* be two parallel profile curves invariably connected wi h a pitch curve K, and let K_r, K_r^* and C, C^* be the rolling curves and drawing points for F and F^* respectively. The roulettc drawn by C, when K_r rolls without sliding along K_r^*, is a circular arc with C^* as its center and the distance of F and F^* as its radius.*

If we donote this circular arc by F_r^*, then by the Camus' theorem in the report (I) the curve F_r^* and F are a pair of profile curves having the curves K_r^* and K as a pair of pitch curves.

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