28. *Fundamental Theory of Toothed Gearing (VI).*

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Suppose that there are given on a unit sphere a pair of pitch curves K_1 and K_2 and a pair of profile curves F_1 and F_2 invariably connected with K_1 and K_2 respectively, and that the length of arcs of K and F are given the signs as well as in the case of plane curves.

§ 1. Sliding of profile curves.

At the sliding contact motion of the profile curves F_1 and F_2, let a part of arc $d\varsigma_1$ of F_1 and a part of arc $d\varsigma_2$ of F_2 slide one along the other during infinitesimal time interval dt, and let $d\xi$ be, in this case, the length of arc of contact of the pitch curve K_1 or K_2. Then the point C on F_1 slides along F_2 for the distance $d\varsigma_2-d\varsigma_1$, and consequently its velocity v_{p1} is given by

$$(1)_1 \qquad v_{p1}=\frac{d\varsigma_2-d\varsigma_1}{dt}.$$

v_{p1} is named the velocity of sliding of F_1 (af the point C on F_2). In like manner the velocity of sliding of F_2 may be defined :

$$(1)_2 \qquad v_{p2}=\frac{d\varsigma_1-d\varsigma_2}{dt}.$$

Denoting by ω_1 and ω_2 respectively the instant angular velocities of K_1 and K_2 at the rolling contact motion and by λ_1 and λ_2 the spherical radii of curvature of K_1 and K_2 respectively at the instant common pitch point P we have

$$(2) \qquad \omega_1=\frac{1}{\sin\lambda_1}\frac{d\xi}{dt}, \quad \omega_2=\frac{1}{\sin\lambda_2}\frac{d\xi}{dt}.$$

Let ω denote the relative rolling angular velocity of K_1 to K_2, then ω is gven by

$$(3) \qquad \omega=\omega_1\cos\lambda_1-\omega_2\cos\lambda_2$$

and accordingly from (2) follows

$$(4) \qquad \omega=\left(\frac{1}{\tan\lambda_1}-\frac{1}{\tan\lambda_2}\right)\frac{d\xi}{dt}.$$

Next, let φ be the singed length of the arc of the great circle connnecting P with the point of contact C of F_1 and F_2, then the velocity v_{p1} of C is represented by $\sin\varphi\cdot\omega$, that is,

$$(5)_1 \qquad v_{p1} = \left(\frac{1}{\tan\lambda_1} - \frac{1}{\tan\lambda_2}\right)\sin\varphi\,\frac{d\xi}{dt}.$$

Similarly

$$(5)_2 \qquad v_{p2} = \left(\frac{1}{\tan\lambda_2} - \frac{1}{\tan\lambda_1}\right)\sin\varphi\,\frac{d\xi}{dt}.$$

From (5) it follows immediately :

Profile curves make rolling contact motion without sliding, if and only if they coincide with their pitch curves.

If, in particular, K_1 and K_2 are both circles and their rotations are of constant velocities, the acceleration of C, w_p, has the tangential and normal component with regard to the profile curve F as follows :

$$(6)_1 \qquad \begin{aligned} w_{t1} &= -\mathrm{sgn}(\varphi)\left(\frac{1}{\tan\lambda_1} - \frac{1}{\tan\lambda_2}\right)\left(\frac{d\xi}{dt}\right)^2 \sin\varphi\cos\theta, \\ w_{n1} &= \left(\frac{1}{\tan\lambda_1} - \frac{1}{\tan\lambda_2}\right)^2\left(\frac{d\xi}{dt}\right)^2 \frac{\sin^2\varphi}{\sin\rho_2}, \end{aligned}$$

and

$$(6)_2 \qquad \begin{aligned} w_{t2} &= -\mathrm{sgn}(\varphi)\left(\frac{1}{\tan\lambda_2} - \frac{1}{\tan\lambda_1}\right)\left(\frac{d\xi}{dt}\right)^2 \sin\varphi\cos\theta, \\ w_{n2} &= \left(\frac{1}{\tan\lambda_2} - \frac{1}{\tan\lambda_1}\right)^2\left(\frac{d\xi}{dt}\right)^2 \frac{\sin^2\varphi}{\sin\rho_1}, \end{aligned}$$

where ρ_1 and ρ_2 denote the spherical radii of curvature of F_1 and F_2 respectively. And further from $(6)_1$ and $(6)_2$ we have

$$(7) \qquad \frac{w_{n1}}{w_{n2}} = \frac{\sin\rho_1}{\sin\rho_2}.$$

Thus we have the following

Theorem 1. *Given a pair of pitch circles which make rolling contact motion with constant velocity of rotation and a pair of profile curves invariably connected with those pitch circles. The velocities of sliding at any point of contact of the profile curves are proportional to the sine of the spherical distance from the point to the pitch point corresponding to it, and the ratio of the normal components of the accelerations is epual to the ratio of the sineo of the spherical radii of curvature of the profile curves.*

When we adopt in particular the rolling curve K_r as one of the pitch curves K_1 and K_2, we have from (5)

$$(8) \qquad \frac{d\varsigma_1}{dt} = \left(\frac{1}{\tan\lambda_r} - \frac{1}{\tan\lambda_1}\right)\sin\varphi\frac{d\xi}{dt}, \quad \frac{d\varsigma_2}{dt} = \left(\frac{1}{\tan\lambda_r} - \frac{1}{\tan\lambda_2}\right)\sin\varphi\frac{d\xi}{dt}.$$

Now the quantities

$$(9) \qquad \sigma_1 = \frac{d\varsigma_2 - d\varsigma_1}{d\varsigma_1}, \quad \sigma_2 = \frac{d\varsigma_1 - d\varsigma_2}{d\varsigma_2}$$

are called respectively the specific slidings of the profile curves F_1 and F_2 (at the point C on F_2 and F_1). Obviously

$$\frac{1}{\sigma_1}+\frac{1}{\sigma_2}=-1. \tag{10}$$

From (1) and (9) we have

$$\sigma_1=v_{p1}\Big/\frac{d\varsigma_1}{dt}, \quad \sigma_2=v_{p2}\Big/\frac{d\varsigma_2}{dt}. \tag{11}$$

Substituting (5) and (9) into (11) we have

$$\sigma_1=\sigma_1(\xi)=\frac{\dfrac{1}{\tan\lambda_1(\xi)}-\dfrac{1}{\tan\lambda_2(\xi)}}{\dfrac{1}{\tan\lambda_r(\xi)}-\dfrac{1}{\tan\lambda_1(\xi)}}, \quad \sigma_2=\sigma_2(\xi)\ \frac{\dfrac{1}{\tan\lambda_2(\xi)}-\dfrac{1}{\tan\lambda_1(\xi)}}{\dfrac{1}{\tan\lambda_r(\xi)}-\dfrac{1}{\tan\lambda_2(\xi)}}. \tag{12}$$

From this follows immediately the fact :

For any profile curves with the same pitch curves and rolling curve the specific slidings at the points of contact corresponding to the same pitch point are all equal, wherever a drawing point is set at the rolling curve.

Furthermore, if the equation $\varphi=f(\xi)$ of the profile curves or the equation $\varphi=g(\theta)$ of the path of contact is given, we can represent the specific slidings in the following forms by substituting Equation (5) or (8) in the report (V) into the above equation (12):

$$\sigma_1=\sigma_1(\xi)=\left(\frac{1}{\tan\lambda_1(\xi)}-\frac{1}{\tan\lambda_2(\xi)}\right)\Big/\left(\frac{1-\{f'(\xi)\}^2-\tan f(\xi)\cdot f''(\xi)}{\tan f(\xi)\sqrt{1-\{f'(\xi)\}^2}}-\frac{1}{\tan\lambda_1(\xi)}\right),$$

(13)

$$\sigma_2=\sigma_2(\xi)=\left(\frac{1}{\tan\lambda_2(\xi)}-\frac{1}{\tan\lambda_1(\xi)}\right)\Big/\left(\frac{1-\{f'(\xi)\}^2-\tan f(\xi)\cdot f''(\xi)}{\tan f(\xi)\sqrt{1-\{f'(\xi)\}^2}}-\frac{1}{\tan\lambda_2(\xi)}\right),$$

or

$$\sigma_1=\sigma_1(\theta)=\frac{\dfrac{1}{\tan\lambda_1(\xi(\theta))}-\dfrac{1}{\tan\lambda_2(\xi(\theta))}}{\dfrac{\sin\theta}{\tan|g(\theta)|}+\dfrac{\cos\theta}{|g(\theta)|}-\dfrac{1}{\tan\lambda_1(\xi(\theta))}},$$

(14)

$$\sigma_2=\sigma_2(\theta)=\frac{\dfrac{1}{\tan\lambda_2(\xi(\theta))}-\dfrac{1}{\tan\lambda_1(\xi(\theta))}}{\dfrac{\sin\theta}{\tan|g(\theta)}+\dfrac{\cos\theta}{|g(\theta)'}-\dfrac{1}{\tan\lambda_2(\xi(\theta))}},$$

where $$\xi(\theta) = -\int \frac{|g(\theta)|'}{\cos\theta} d\theta.$$

§ 2. The types and the radii of curvature of profile curves.

In consequence of (8) we have the following relations (15) between the arc lenth $d\xi$ of the pitch curve K and the arc length $d\xi$ of the profile curve F corresponding to it :

$$(15) \qquad d\varsigma_1 = \left(\frac{1}{\tan\lambda_r} - \frac{1}{\tan\lambda_1}\right)\sin\varphi\, d\xi, \quad d\varsigma_2 = \left(\frac{1}{\tan\lambda_r} - \frac{1}{\tan\lambda_2}\right)\sin\varphi\, d\xi.$$

In accordance with (15) we can derive the following theorem concerning the types of roulettes, namely, of profile curves.

Theorem 2. *Let a curve K_r with the natural equation $\lambda_r = \lambda_r(\xi)$ roll without sliding along a curve K with the natural equation $\lambda = \lambda(\xi)$ In the range of ξ, where the spherical curvature $\frac{1}{\lambda_r(\xi)}$ of K_r is larger than K' s ; $\frac{1}{\lambda_r(\xi)} > \frac{1}{\lambda(\xi)}$, the roulette F drawn by a point C fixed at K_r is of positive type as far as the point C exists on the left side of the common tangent great circle of K_r and K at the common pitch point, and of negative type as far as C exists on the right side. In the range, where $\frac{1}{\lambda_r(\xi)} < \frac{1}{\lambda(\xi)}$, the converse holds.*

Moreover, we have the following theorem concerning the assortion of the types of a pair of profile curves.

Theorem 3. *Let the natural equations of a pair of profile curves K_1 and K_2 and rolling curve K_r be $\lambda_1 = \lambda_1(\xi)$, $\lambda_2 = \lambda_2(\xi)$ and $\lambda_r = \lambda_r(\xi)$ respectively. In the range of ξ, where the spherical curvature $\frac{1}{\lambda_r(\xi)}$ of K_r is larger or smaller than both of the spherical curvatures $\frac{1}{\lambda_1(\xi)}$ and $\frac{1}{\lambda_2(\xi)}$ of K_1 aud K_2, in other words, both of K_1 and K_2 exist on one side of K_r, the same type parts of F_1 and F_2 are in mesh, and in the range, where $\frac{1}{\lambda_r(\xi)}$ exists between $\frac{1}{\lambda_1(\xi)}$ and $\frac{1}{\lambda_2(\xi)}$ that is, K_1 exists between K_1 and K_2, the different type parts of F_1 and F_2 are in mesh.*

By theorem 10 in the report (V), λ, the spherical radius of curvature of K_r is equal to the length of the segment cutten off by the normal great circle to the path of contact Γ on the perpendicular great circle $P_0 N_0$ to the initial line $P_0 T_0$ at the pole P_0. Consequently, when both of the pitch curves K_1 and K_2 are small circles, we can state Theorem 3 in the following manner.

Theorem 4. *Given a pair of pitch circles O_1 and O_2 touching at a point P_0 and a path of contact Γ settled at their common tangent great circle P_0T_0. Let M be the point at which a normal great circle to Γ intersects the great circle $O_1 O_2$ connecting the centers of the* *circles O_1 and O_2. As far ar M exists on the one of the two parts of the center line O_1O_2 devided by the two points O_1 and O_2, on which part the point P_0 is contained, the same type parts of the profile curves corresponding to Γ are in mash. When M exists on the part not containing P_0, the different type parts of the profile curves are in mesh.*

We denote by ρ the spherical radius of curvature of the profile curve F at the point C on F. As we have defined, the infinitesimal arc $d\varsigma$ of F is oriented, according to this orientation we give ρ a positive or negative sign.

Then we have

$$(16)\qquad \frac{d\varsigma}{d\xi}=\frac{\sin(\pm\rho)}{\sin(\varphi\pm\rho)}\,|\sin\theta|,$$

where we take, from the double signs $\pm$ before ρ, + if F is of positive type and − if F of negative type. It follows from (15) and (16)

$$(17)\qquad \frac{1}{\tan\lambda_r}-\frac{1}{\tan\lambda}=\left(\frac{1}{\tan\varphi}-\frac{1}{\tan(\varphi\pm\rho)}\right)|\sin\theta|.$$

This is the generalization to the spherical curves of the formula of Savary concerning the radius of curvature of a roulette drawn by a point fixed at a curve K_r when K_r rolls without sliding along a curve K.

From (17) we can derive the relation between the spherical radii of curvature ρ_1 and ρ_2 of a pair of profile curve F_1 and F_2 at a point of contact :

$$(18)\qquad \frac{1}{\tan\lambda_1}-\frac{1}{\tan\lambda_2}=\left(\frac{1}{\tan(\varphi\pm\rho_1)}-\frac{1}{\tan(\varphi\pm\rho_2)}\right)|\sin\theta|,$$

where out of the double signs before ρ_1 and ρ_2— in total four signs—, we assort the same two if F_1 and F_2 are of the same type, and the different two if F_1 and F_2 are of the different types.

Substituting (17) and (18) into (12) we have

$$(19)\qquad \sigma_1=\frac{\dfrac{1}{\tan(\varphi\pm\rho_1)}-\dfrac{1}{\tan(\varphi\pm\rho_2)}}{\dfrac{1}{\tan\varphi}-\dfrac{1}{\tan(\varphi\pm\rho_1)}},\quad \sigma_2=\frac{\dfrac{1}{\tan(\varphi\pm\rho_2)}-\dfrac{1}{\tan(\varphi\pm\rho_1)}}{\dfrac{1}{\tan\varphi}-\dfrac{1}{\tan(\varphi\pm\rho_2)}}.$$

Now, we shall consider a profile curve F and a parallel profile curve F^* with the spherical distance α from F. Denote by K_r and K_r^* respectively the rolling curves for F and F^* and let $\lambda_r=\lambda_r(\xi)$ and $\lambda_r^*=\lambda_r^*(\xi)$ be the natural equation of K_r and K_r^* respectively. By Equation (4) in the report (V) we can derive

$$(20)\qquad \frac{1}{\tan\lambda_r^*}-\frac{1}{\tan\lambda_r}=\left(\frac{1}{\tan(\varphi+\alpha)}-\frac{1}{\tan\varphi}\right)|\sin\theta|.$$

Comparing (20) with (17) we obtain the following

Theorem 5. *Let F and F^* be two parallel profile curves invariably connected with a pitch curve K, and let K_r, K_r^* and C, C^* be the rolling curves and drawing points for F and F^* respectively. The roulette drawn by C when K_r rolls without sliding along K_r^* is a circular arc with C^* as its center and the spherical distance of F and F^* as its spherical radius.*

If we denote this circular arc by F_r^*, then the curve F_r^* and F are a pair of profile curves having the curves K_r^* and K as a pair of pitch curves.

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