78. An Alternative Proof of a Generalized Principal Ideal Theorem.

By Tadao TANNAKA.

Mathematical Institute, Tohoku University, Sendai.

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Recentry Mr. Terada[1] has proved the following generalized principal theorem :

Theorem. Let K be the absolute class field over k, and Ω a cycic intermediate field of K/k, then all the ambigous ideal classes of Ω will become principal in K.

I also generalized this theorem to the case of ray class field.[2]

By using Artin's law of reciprocity we can state above theorem in terms of the Galois group, and we have

Theorem. Let G be a metabelian group with commutator subgroup G', H be an invariant subgroup of G with the cyclic quotient group G/H, and A element of H with $ASA^{-1}S^{-1}\epsilon H'$ (S being a generator of G/H), then the "Verlagerung" $V(A) = \prod TA\overline{TA}^{-1}$ from H to G' is the unit element of G. Thereby T runs over a fixed representative system of G/H, and $\overline{TA}$ means the representative corresponding to the coset $\overline{TA}G'$.

At first we tried to solve this by means of Iyanaga's method depending upon Artin's splitting group,[3] which is generated by G' and the symbols $A_\sigma (A_1 = 1,\ \sigma \epsilon \Gamma = G/G')$, and with Γ as operator system by rules

(1) $$U^\sigma = S_\sigma U S_\sigma^{-1} \quad (U \epsilon G'),$$

(2) $$A_\tau^\sigma = A_\sigma^{-1} A_{\sigma\tau} D_{\sigma,\tau}^{-1},$$

S_σ being the representative of G/G' corresponding to $\sigma \epsilon \Gamma$, and

(3) $$D_{\sigma,\tau} = S_\sigma S_\tau S_{\sigma\tau}^{-1}.$$

But it seemed to us as if his method were not so easily applicable to our problem, and Terada at last checked the classical method of Furtwängler,[4] which brought him to success, after a rather complicated computation.

Here I will give a more simple proof, which depends upon Artin's splitting group.

We first transform the problem into additive form (cf. Terada[1]) and then by using Artin's splitting group (as in Tannaka[2]) our theorem is reduced to a proposition concerning the additive group with a commutative ring R as operator domain. Thus we now prove the

Theorem. Let M be an additive group with the (not necessarily independent) base elements $c_1, c_2, \ldots, c_n$ and c, and

$$N_i c_i = \sum_{r,s=1}^{n} A_{r,s}^{(i)} \Delta_r c_s + \sum_{j=1}^{n} B_j^{(i)} \delta_j \quad (i = 1, 2, \ldots, n) \tag{1}$$

where

$$\delta_i = \Delta c_i - \Delta_i c, \tag{2}$$

$$N_i \Delta_i = 0, \tag{3}$$

$$A_{r,s}^{(i)} = -A_{s,r}^{(i)}, \quad A_{r,r}^{(i)} = 0. \tag{4}$$

If then

$$\sum \Gamma_i \delta_i = \sum F_{r,s} \Delta_r c_s \ (F_{r,s} = -F_{s,r}, \ F_{r,r} = 0), \tag{5}$$

we have

$$N_1 \ldots N_n \sum \Gamma_i c_i = 0. \tag{6}$$

N.B. Concerning the meaning of notations we refer to Terada[1] and Tannaka[2]. Δ represents the operator $1\text{-}S$, and c is an element in M, which corresponds to A_s in the splitting group. $\varepsilon_{ij} = \Delta_j c_i - \Delta_i c_j$ corresponds to the commutator of the form $S_i S_j S_i^{-1} S_j^{-1}$.

Proof. Form (1) and (2) we have

$$\sum_{j=1}^{n} (N_i \delta_{ij} - \sum_r A_{rj}^{(i)} \Delta_r - B_j^{(i)} \Delta) c_j = -\sum_j B_j^{(i)} \Delta_j c, \tag{1'}$$

$$(\delta_{ii} = 1, \ \delta_{ij} = 0, \ i \neq j; \ i = 1, 2, \ldots, n),$$

therefore by Cramér's method of elimination

$$\begin{vmatrix} N_1 + \sum_r A_{1r}^{(1)} \Delta_r - \Delta B_1^{(1)}, & \ldots, & \sum_r A_{nr}^{(1)} \Delta_r - \Delta B_n^{(1)} \\ \ldots & \ldots & \ldots \\ \sum_r A_{1r}^{(n)} \Delta_r - \Delta B_1^{(n)}, & \ldots, & N_n + \sum_r A_{nr}^{(n)} \Delta_r - \Delta B_n^{(n)} \end{vmatrix} c_n \tag{7}$$

$$= \begin{vmatrix} N_1 + \sum_r A_{1r}^{(1)} \Delta_r - \Delta B_1^{(1)}, & \ldots, & -\sum_j B_j^{(1)} \Delta_j \\ \ldots & \ldots & \ldots \\ \sum_r A_{1r}^{(n)} \Delta_r - \Delta B_1^{(n)}, & \ldots, & -\sum_j B_j^{(n)} \Delta_j \end{vmatrix} c,$$

or

(8)
$$\begin{vmatrix} a_{11}, \ldots, a_{1n} \\ \cdots\cdots\cdots \\ a_{n1}, \ldots, a_{nn} \end{vmatrix} c_n = \begin{vmatrix} a_{11}, \ldots, a_{1,n-1}, \beta_1 \\ \cdots\cdots\cdots\cdots \\ a_{n1}, \ldots, a_{n,n-1}, \beta_n \end{vmatrix} c,$$

if we put

(9)
$$\begin{cases} a_{ij} = N_i\delta_{ij} + \sum_r A_{jr}^{(i)}\Delta_r - B_j^{(i)}\Delta, \\ \beta_i = -\sum_j B_j^{(i)}\Delta_j. \end{cases}$$

We put further

(10)
$$D = |a_{ij}| - N_1 \ldots N_n.$$

In my preceding paper[2] I obtained the identity

(11)
$$|\sum_r A_{jr}^{(i)}\Delta_r| = 0$$

and as its consequence

(12)
$$|N_i\delta_{ij} + \sum_r A_{jr}^{(i)}\Delta_r| = N_1 \ldots N_n,$$

so that in the expansion of D, every term has Δ as a faetor.

We now deduce the fundamental relation:

(13)
$$N_1 \ldots N_n c_i = -\frac{D}{\Delta}\delta_i \quad (i = 1, 2, \ldots, n).$$

In Terada's paper this formula is given in the expanded form, consequently it was somewhat complicated. Anyhow (13) was the key point of his success.

Without loss of generality we can restrict ourselves to the case $i = n$ in (13), so that we have only to prove

(13')
$$N_1 \ldots N_n c_n = -\frac{D}{\Delta}\delta_n.$$

From (8) and $\delta_n = \Delta c_n - \Delta_n c$ we have

$$N_1 \ldots N_n c_n + Dc_n = N_1 \ldots N_n c_n + \frac{D}{\Delta}\delta_n + \frac{D}{\Delta}\Delta_n c$$

$$= \begin{vmatrix} a_{11}, \ldots a_{1,n-1}\ \beta_1 \\ a_{n1} \ldots a_{n,n-1}\ \beta_n \end{vmatrix} c,$$

so that (13') may be reduced to

(14)
$$\frac{D}{\Delta}\Delta_n = \begin{vmatrix} a_{11} \ldots a_{1\,n-1}\ \beta_1 \\ \\ a_{n1} \ldots a_{n\,n-1}\ \beta_n \end{vmatrix}.$$

First we define D by

$$(15) \qquad D_n = \begin{vmatrix} a_{11} \cdots\cdots a_{1\,n-1}\; a_1'' \\ \cdots\cdots\cdots\cdots \\ a_{n1} \cdots\cdots a_{n\,n-1}\; a_n'' \end{vmatrix} \quad \left(a_i'' = \sum_r A_{nr}^{(i)} \Delta_r\right.$$

and prove

$$(16) \qquad \frac{D_n}{\Delta} \Delta_n = \frac{D}{\Delta} \Delta_n .$$

We have indeed

$$\frac{D}{\Delta}\Delta_n - \frac{D_n}{\Delta}\Delta_n = \left\{ \begin{vmatrix} a_{11} \cdots\cdots\cdots a_{1,\,n-1} \\ \cdots\cdots\cdots\cdots \\ a_{n-1,\,1} \cdots\cdots a_{n-1,\,n-1} \end{vmatrix} N_n - N_1 \cdots\cdots N_n \right\} \frac{\Delta_n}{\Delta}$$

and as we have $\Delta_n N_n = 0$ and

$$\left| N_i \delta_{ij} + \sum_r A_{jr}^{(i)} \Delta_r \right|_{i,j \leq n-1} N_n$$

$$= \left| N_i \delta_{ij} + \sum_{r \leq n-1} A_{jr}^{(i)} \Delta_r \right|_{i,j \leq n-1} N_n = (N_1 \cdots\cdots N_{n-1}) N_n$$

by (12). above expression reduces to 0.

Now we prove the equality

$$(17) \qquad \frac{D_n}{\Delta} \Delta_n = \begin{vmatrix} a_{11} \cdots\cdots a_{1\,n-1}\; \beta_1 \\ \cdots\cdots\cdots\cdots \\ a_{n1} \cdots\cdots a_{n\,n-1}\; \beta_n \end{vmatrix}$$

by induction on n.

If we expand both members of (17) in terms of N_i, and calling the relations $N_i \Delta_i = 0$ in mind, we see that the terms with N_i as factor cancel out each other, for instance the terms with N_1 as coefficient in the first member is

$$N_1 \left| N_j \delta_{ij} + \sum A_{jr}^{(i)} \Delta_r \right|_{i,j \geq 2} \frac{\Delta_n}{\Delta}$$

$$= N_1 \left| N_j \delta_{ij} + \sum_{r \geq 2} A_{jr}^{(i)} \Delta_r \right|_{i,j \geq 2} \frac{\Delta_n}{\Delta}$$

(under the convention that in the last column of $\left| N_j \delta_{ij} + \sum_{r \geq 2} A_{jr}^{(i)} \Delta_r \right|_{i,j \geq 2}$, $N_j \delta_{ij} + \sum_{r \geq 2} A_{jr}^{(i)} \Delta_r$ should be replaced by $\sum_{r \geq 2} A_{jr}^{(i)} \Delta_r$), so that the coefficient of N_1 is of the same form as that of the first member of (17), except for the degree of determinant. So we have to prove only

$$
(18) \qquad \begin{vmatrix} \alpha'_{11} & \dots & \alpha'_{1n} \\ & \dots\dots & \\ \alpha'_{n1} & \dots & \alpha'_{nn} \end{vmatrix} \frac{\Delta_n}{\Delta} = \begin{vmatrix} \alpha'_{11} & \dots & \alpha'_{1\,n-1} & \beta_1 \\ & \dots\dots & & \\ \alpha'_{n1} & \dots & \alpha'_{n\,n-1} & \beta_n \end{vmatrix},
$$

$$
(\alpha'_{ij} = \sum_r A^{(i)}_{jr}\Delta_r - B^{(i)}_j\Delta,\ \beta_i = -\sum_r B^{(i)}_r\Delta_r).
$$

Expanding the both members, there remain only the terms with some $B^{(i)}_j$ as factor, for in the first member $|\sum_r A^{(i)}_{jr}\Delta_r| = 0$ is the total contribution of such term. Hence we compare for instance the terms with $B^{(1)}_1$ as factor, and prove

$$
-B^{(1)}_1\Delta_n\,|\alpha'_{ij}|_{i,j\geq 2}
$$

$$
= -B^{(1)}_1\Delta \begin{vmatrix} \alpha'_{22} & \dots & \alpha'_{2\,n-1} & \beta_2 \\ & \dots\dots & & \\ \alpha'_{n2} & \dots & \alpha'_{n\,n-1} & \beta_n \end{vmatrix} + (-1)^n B^{(1)}_1\Delta_1 \begin{vmatrix} \alpha'_{21} & \dots & \alpha'_{2\,n-1} \\ & \dots\dots & \\ \alpha'_{n1} & \dots & \alpha'_{n\,n-1} \end{vmatrix}
$$

$$
\equiv -B^{(n)}_1\Delta \begin{vmatrix} \alpha'_{22} & \dots & \alpha'_{2\,n-1} & \beta_2 \\ & \dots\dots & & \\ \alpha'_{n2} & \dots & \alpha'_{n\,n-1} & \beta_n \end{vmatrix} - B^{(1)}_1\Delta_1 \begin{vmatrix} \alpha'_{22} & \dots & \alpha'_{2\,n-1} & \alpha'_{21} \\ & \dots\dots & & \\ \alpha'_{n2} & \dots & \alpha'_{n\,n-1} & \alpha'_{n1} \end{vmatrix},
$$

or

$$
(19) \qquad \begin{vmatrix} \alpha'_{22} & \dots & \alpha'_{2\,n-1} & \gamma_2 \\ & \dots\dots & & \\ \alpha'_{n2} & \dots & \alpha'_{n\,n-1} & \gamma_n \end{vmatrix} = 0,
$$

where

$$
\gamma_i = \alpha'_{in}\Delta_n - \Delta\beta_i + \alpha'_{i1}\Delta_1
$$

$$
= (\sum_r A^{(i)}_{nr}\Delta_r - B^{(i)}_n\Delta)\Delta_n + \Delta\sum_r B^{(i)}_r\Delta_r + (\sum_r A^{(i)}_{1r}\Delta_r - B^{(i)}_1\Delta)\Delta_1
$$

$$
= \sum_{s=1,n}\sum_r A^{(i)}_{sr}\Delta_s\Delta_r + \Delta\sum_{1<r<n} B^{(i)}_r\Delta_r.
$$

But as

$$
\alpha'_{i2} + \dots + \alpha'_{i,\,n-1} + \gamma_i = \sum_{s,r} A^{(i)}_{sr}\Delta_s\Delta_r = 0
$$

we have established (19), and the proof of (16) is completed.

We now proceed to the second part of our proof, that is, the proof of (6).

As it holds $N_1 \dots N_n \sum \Gamma_i c_i = \sum \Gamma_i N_1 \dots N_n c_i = -\sum \Gamma_i \frac{D}{\Delta}\delta_i$ by (13) and

$$
= -\frac{D}{\Delta}\sum F_{r,s}\Delta_r c_s
$$

by (5), it suffices to prove for instance the equality

$$\frac{D}{\Delta}\,\varepsilon_{12}=\frac{D}{\Delta}\,(\Delta_2 c_1-\Delta_1 c_2)=0\,. \tag{20}$$

But it follows from (16) and (17)

$$\frac{D}{\Delta}\,\varepsilon_{12}=\frac{D_2}{\Delta}\,\Delta_2 c_1-\frac{D_1}{\Delta}\,\Delta_1 c_2$$

$$=\begin{vmatrix} a_{11}\beta_1 a_{13}\ldots\ldots a_{1n} \\ \ldots\ldots\ldots\ldots \\ a_{n1}\beta_n a_{n3}\ldots\ldots a_{nn}\end{vmatrix} c_1-\begin{vmatrix} \beta_1 a_{12}\ldots\ldots a_{1n} \\ \ldots\ldots\ldots \\ \beta_n a_{n2}\ldots\ldots a_{nn}\end{vmatrix} c_2$$

$$=\begin{vmatrix} a_{11}c_1+a_{12}c_2,\ \beta_1,\ a_{13},\ \ldots\ldots,\ a_{1n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}c_1+a_{n2}c_2,\ \beta_n,\ a_{n3},\ \ldots\ldots,\ a_{nn}\end{vmatrix}$$

$$=\begin{vmatrix} \sum a_{1r}c_r+\beta_1 c,\ \beta_1,\ a_{13},\ \ldots\ldots\ a_{1n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \sum a_{nr}c_r+\beta_n c,\ \beta_n,\ a_{n3},\ \ldots\ldots\ a_{nn}\end{vmatrix},$$

and as $\sum_r a_{ir}c_r+\beta_i c=0$ by (14), we have $\frac{D}{\Delta}\,\varepsilon_{12}=0$ as desired, q.e.d.

References

1) F. Terada: On the generalization of the principal ideal theorem, Tôhoku Math. J., (2) 1, No. 2 (1949).

2) T. Tannaka: Some remarks concerning principal ideal theorem, Tôhoku Math. J., (2) 1, No. 2 (1949).

3) S. Iyanaga: Zum Beweis des Hauptidealsatzes. Hamb. Abh., 10 (1934).

4) Ph. Furtwängler: Beweis des Hauptidealsatzes, Hamb. Abh., 7 (1930).

Concerning 1) and 2) we also refer to the previous note, titled "A generalization of principal ideal theorem" in Proc. Acad. Tokyo (1949).