62. Note on Pseudo-Analytic Functions.

By Eiichi SAKAI.

Mathematical Institute, Nagoya University.

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1. Let $w=f(z)=u(x, y)+iv(x, y)$, $z=x+iy$, be an inner transformation in the sense of Stoïlow in a connected domain D. Denote by E a set, in D, such that D and the derived set E' of E have no point in common. We suppose that

a) u_x, u_y, v_x, v_y exist and are continuous in $D^*=D-E$,

b) $J(z)=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} > 0$ at every point in D^*,

c) the function $q(z)$ defined as the ratio of major and minor axes of an infinitesimal ellipse with centre $f(z)$, into which an infinitesimal circle with centre at every point z of D^* is transformed by $f(z)$, is bounded in $D^*: q(z) \leqq K$.

$f(z)$ is then called pseudo-meromorphic (K) in D[1].

The purpose of the present note is to give some results concerning pseudo-conformal representations and the cluster sets of pseudo-meromorphic functions.

2. Let $w=f(z)$ be pseudo-meromorphic (K) in a connected domain D. It is known that the set of $[z, w]$ where $w=f(z)$, $z \in D$, defines a Riemann surface Φ, in the sense of Stoïlow, spread over the w-plane. By the theory of uniformizations of P. Koebe, there exists a function $z'=\varphi(w)$ analytic in Φ which maps Φ on a plane (*schlicht*) domain D' of the z'-plane. Consequently we get a function $z(z')$ (or $z'(z)$) which defines a pseudo-conformal mapping (K) between D and D', by eliminating w from $w=f(z)$ and $z'=\varphi(w)$.

Thus we see that a function $w=f(z)$, pseudo-meromorphic (K) in D, is a composition of a uniform function $w=\varphi^{-1}(z')$, analytic in D' and a univalent function $z'(z)$, pseudo-regular (K) in D.

In view of the above consideration, it may be of some interest to investigate "*Verzerrungssatz*" concerning pseudo-conformal mapping (K). We first show that the properties of Fatou and Gross-Ahlfors hold for a bounded and univalent function, pseudo-regular (K).

Theorem 1. (Fatou's property). *Let* $w=f(z)=u(r, \theta)+iv(r, \theta)$, $z=re^{i\theta}$, *be*

1) S. Kakutani, *Applications of the theory of pseudo-regular functions to the type-problem of Riemann surfaces*, Jap. Journ. of Math., vol. 13 (1937), pp. 375–392.

bounded, univalent and pseudo-regular (K) *in* $|z|<1$. *Then there exists the radial limit* $\lim_{r\to 1} f(re^{i\theta})=f(e^{i\theta})$ *for almost all values of* θ[2].

Proof. Consider the function

$$(1) \qquad F(r,\theta)\equiv\int_{r_0}^{r} ds=\int_{r_0}^{r}\frac{ds}{|dz|}\cdot|dz| \qquad (z=re^{i\theta},\ 0<r_0<r<1),$$

which is measurable in Lebesgue's sense for $0\leqq\theta\leqq 2\pi$, where

$$\frac{ds}{|dz|}=\frac{ds}{dr}=\left|\frac{\partial u(r,\theta)}{\partial r}+i\frac{\partial v(r,\theta)}{\partial r}\right|.$$

Applying Schwarz' inequality, we have from (1)

$$[F(r,\theta)]^2\leqq\int_{r_0}^{r}\left(\frac{ds}{|dz|}\right)^2 dr\cdot\int_{r_0}^{r} dr$$

$$=(r-r_0)\int_{r_0}^{r}\left(\frac{ds}{|dz|}\right)^2 dr^{3)}$$

$$\leqq(r-r_0)\int_{r_0}^{r}[q(z)+\sqrt{q(z)^2-1}]\cdot J(z)dr.$$

By the assumption $q(z)\leqq K$, it follows that

$$[F(r,\theta)]^2\leqq 2K\cdot\frac{r-r_0}{r_0}\int_{r_0}^{r} J(z)rdr.$$

Integrating the both sides with respect to θ from 0 to 2π, we get

$$\int_0^{2\pi}[F(r,\theta)]^2 d\theta\leqq 2K\cdot\frac{1-r_0}{r_0}\int_0^{2\pi}\int_{r_0}^{1} J(z)rdrd\theta.$$

Since the double integral of the right member is the area of the image of $r_0<|z|<1$ by $w=f(z)$,

$$(2) \qquad \int_0^{2\pi}[F(r,\theta)]^2 d\theta<M,$$

where M is a fixed constant independent of r.

Now, $F(r,\theta)$ is a non-negative and measurable function of θ and is monotone increasing as $r\to 1$. Consequently

$$\int_0^{2\pi}[F(\theta)]^2 d\theta=\lim_{r\to 1}\int_0^{2\pi}[F(r,\theta)]^2 d\theta<M,$$

where $F(\theta)\equiv\lim_{r\to 1}F(r,\theta)$. Hence $F(\theta)\equiv\int_{r_0}^{1} ds$ is finite for almost all values of θ; so that the curves corresponding to all radii $z=re^{i\theta}(r_0\leqq r\leqq 1,\ 0\leqq\theta\leqq 2\pi)$ save exceptional values of θ on the w-plane are rectifiable. In other words, $f(z)$ have radial limits along almost all radii.

Next, we shall state a theorem which will be used as a lemma in the following paragraph.

Theorem 2. (Gross-Ahlfors' property). *Suppose that* $w=f(z)$ *is univalent*

2) This extension of Fatou's theorem was given by Prof. K. Noshiro.
3) See for instance R. Nevanlinna: *Eindeutige analytische Funktionen* (1936), p. 344.

bounded and pseudo-regular (K) *in an open set* D *and let* z_0 *be a boundary point of* D. *Then, denoting by* $L(r)$ *the total length of the image of* θ_r *by* $w=f(z)$, *where* θ_r *is the common part of* $|z-z_0|=r$ *and* D, *we have* $\underline{\lim}_{r\to 0} L(r)=0$.

Proof. By Ahlfors' method, as in the proof of the preceding theorem, we have

$$[L(r)]^2 \leqq 2\pi r \int_{0_r} [q(z)+\sqrt{q(z)^2-1}\,] \cdot J(z)\,|dz| \qquad (z=z_0+re^{i\theta},\ q(z)\leqq K),$$

whence

$$\frac{[L(r)]^2}{4K\pi r} \leqq \int_{0_r} J(z) \cdot r d\theta.$$

If we had $\underline{\lim}_{r\to 0} L(r)>0$, then there would exist two positive numbers r_0 and δ such that $L(r)>\delta>0$ for $0<r\leqq r_0$. Accordingly

$$\frac{\delta^2}{4K\pi}\int_r^{r_0}\frac{dr}{r}<\int_r^{r_0}\int_{0_r} J(z)\cdot r dr d\theta,$$

whence

$$\frac{\delta^2}{4K\pi}\log\frac{r_0}{r}<A(r_0),$$

where $A(r_0)$ denotes the area of the image of the intersection of $|z-z_0|<r_0$ and D by $w=f(z)$. Thus we would arrive at a contradiction, making r tend to zero.

Remark. The theorem holds good even if $\lim_{r\to 0}\int_r^{r_0}\frac{dr}{rq(r)}$ diverges where $q(r)=\max_{|z-z_0|=r} q(z)$. Further, taking as $L(r)$ the total spherical length of the image of θ_r by $w=f(z)$, we may obtain the relation $\underline{\lim}_{r\to 0} L(r)=0$ under the assumption that $f(z)$ is p-valent in D. Using the line-element $d\tau$ and the area-element $d\omega$ on the Riemann sphere given by

$$d\tau=\frac{|dw|}{1+|w|^2} \quad\text{and}\quad d\omega=\frac{dudv}{(1+|w|^2)^2}=\frac{J(z)dxdy}{(1+|w|^2)^2}$$

respectively, we can enunciate the theorem in the following form;

Suppose that $w=f(z)$ *is* p*-valent and pseudo-meromorphic* (K) *in an open set* D *and let* z_0 *be a boundary point of* D. *Then, denoting by* $L(r)$ *the total spherical length of the image of* θ_r *by* $w=f(z)$, *where* θ_r *is the common part of* $|z-z_0|=r$ *and* D, *we have* $\underline{\lim}_{r\to 0} L(r)=0$.

We see that various theorems in the theory of conformal representation hold true if the mapping is pseudo-conformal (K), using the theorems and arguments similar as in M. Tsuji's paper[4]. For example, we can prove that

4) M. Tsuji, *On the théorems of Carathéodory and Lindelöf in the theory of conformal representation*, Jap. Journ. of Math., vol. 7 (1930), pp. 91–99.

Carathéodory's theorem concerning Jordan domain: "If $w=f(z)$ maps a Jordan domain D conformally upon $|z|<1$, the correspondence between $|z|<1$ and D is of a one-to-one and bicontinuous manner their boundaries included," holds even if the mapping is pseudo-conformal (K).

3. Using Theorem 2, we shall enunciate that the results of K. Noshiro[5] hold for the case where $w=f(z)$ is pseudo-meromorphic (K).

Theorem 3. *Let $w=f(z)$ be pseudo-meromorphic (K) in an arbitrary domain D. Suppose that α is a value belonging to $S_{z_0}^{(D)}-S_{z_0}^{(C)}$ but not belonging to $R_{z_0}^{(C)}$,[6] where z_0 is a point on the boundary C of D. Then z_0 becomes necessarily accessible and α is an asymptotic value of $f(z)$ at z_0.*

Proof. Since the case where z_0 is an isolated boundary point of D is obvious, we have only to consider the case where z_0 is non-isolated (whence, evidently, $S_{z_0}^{(C)}\neq 0$) and α is finite, for brevity. Then, as in the proof[7] of the corresponding theorem of K. Noshiro, there are two positive numbers r and ρ such that $f(z)-\alpha\neq 0$ for $|z-z_0|\leqq r$ inside D and further both

$$\overline{\bigcup_{0\leqq|\xi-z_0|\leqq r} S_{\xi}^{(D)}}\ (\xi\in C) \quad \text{and} \quad \overline{\bigcup_{z\in(|z-z_0|=r)\cap D} f(z)}$$

lie outside the circle $(c):|w-\alpha|<\rho$. On the other hand, as α is a cluster value of $f(z)$ at z_0, we may find a suitable point ζ_0 in $D_r\equiv(|z-z_0|<r)\cap D$, whose image $w_0=f(\zeta_0)$ lies within (c). Suppose that ζ_0 is contained in the component D_0 of D_r. Then $f(z)$ is pseudo-meromorphic (K) and $f(z)\neq\alpha$ in D_0,

$$|f(\zeta_0)-\alpha|<\rho \quad \text{and} \quad S_{\xi}^{(D_0)}(0<|\xi-z_0|\leqq r,\ \xi\in C)$$

lies outside (c).

Now we consider the function $\omega=\psi(z)=\dfrac{1}{f(z)-\alpha}$, pseudo-regular (K) in D_0, whose cluster values at each boundary point of D_0, distinct from z_0, lie all

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5) K. Noshiro, *On the theory of the cluster sets of analytic functions*, Journ. Fac. of Sci. Hokkaido Imp. Univ. (1) **6**, No. 4 (1938), pp. 217–231.

6) We associate with z_0 the following three sets of values:

(1) The cluster set $S_{z_0}^{(D)}$. This is the set of all values α such that $\lim_{\nu\to\infty} f(z_\nu)=\alpha$ with a sequance $\{z_\nu\}$ of points tending to z_0 inside D. In other words, $S_{z_0}^{(D)}$ is identical with the intersection $\bigcap_r \overline{\mathfrak{D}}_r$, where $\overline{\mathfrak{D}}_r$ is the closure of the set $\mathfrak{D}_r$ of values taken by $w=f(z)$ inside the common part of $|z-z_0|<r$ and D.

(2) The cluster set $S_{z_0}^{(C)}$. This is the intersection $\bigcap_r M_r$, where M_r denotes the closure of the union $\bigcup_{\xi} S_{\xi}^{(D)}$ for all ξ belonging to the common part of C and $0<|z-z_0|<r$.

(3) The range of values $R_{z_0}^{(D)}$. This is the product of all value-sets $\mathfrak{D}_r$ of $f(z)$ for $(|z-z_0|<r)\cap D$.

7) See 5), pp. 218–219.

inside $|w|<\frac{1}{\rho}$ and such that $|\psi'(\zeta_0)|>\frac{1}{\rho}$. Consider the Riemannian image Φ_0 of D_0 by $\omega=\psi'(z)$ and put $\psi'(\zeta_0)=\omega_0$. Then we make a star-region with respect to ω_0 in W, where W is the complement of the angular domain bounded by two tangents from ω_0 of the circle $|\omega|<\frac{1}{\rho}$. Next, consider the star-region in Gross' sense formed by the sum of segments from ω_0 to irregular points along along all half-lines $\arg(\omega-\omega_0)=\theta$ on Φ_0 whose projections lie inside W. We shall show that the linear measure of the set of amplitudes of singular rays (by which we understand these half-lines meeting at a singular points in a finite distance) must be equal to zero. Since there is at most an enumerable number of algebraic singularities on Φ_0, it is sufficient to show that the set of amplitudes of the singular rays with end points at transcendentral singularities is of linear measure zero. We must have the curve on the z-plane, joining the point ζ_0, lying inside D_0, to z_0, whose image is the segment with the end point at a transcendental singularity. On the other hand, as in the proof of the theorem (*Sternsatz*) of Gross, it follows from Theorem 2 that the linear measure of the preceding set is equal to zero. Hence, almost all rays starting from ω_0 in W grow up to ∞. Since obviously their counter-images must always lie inside D_0, our theorem is proved.

Applying Theorems 2 (see Remark) and 3, we may give by a discussion similar to the proof[8] of the corresponding theorem of Noshiro's paper the following

Theorem 4. *Suppose that $w=f(z)$ is pseudo-meromorphic (K) in an arbitrary domain D and let z_0 be a non-isolated point on the boundary C of D. Suppose further that $f(z)$ is at most p-valent near z_0 inside D. Then we have the relation* $S_{z_0}^{(D)}=S_{z_0}^{(C)}$.

Remark. As is well known, Theorem 4 has been improved in the case where $w=f(z)$ is a uniform function meromorphic in D. It seems difficult for me to prove the following theorem, when $f(z)$ is pseudo-meromorphic (K) in D:

Theorem I. (The first property of Beurling-Kunugui)[9] *Under the assumptions of the first half of Theorem 4, we have* $B(S_{z_0}^{(D)})\subset S_{z_0}^{(C)}$, *where* $B(S_{z_0}^{(D)})$ *denotes the boundary of* $S_{z_0}^{(D)}$, *or, what is the same,* $\Omega=S_{z_0}^{(D)}-S_{z_0}^{(C)}$ *is an open set.*

8) See 5), pp. 222–223.

9) K. Kunugui, *Sur un théorème de MM. Seidel-Beurling*, Proc. Imp. Acad. Tokyo, **15** (1939), pp. 27–32.

However the theorem may probably be true. Now, if Theorem I be true, the second theorem of Beurling-Kunugui[10] may also hold true in our case without modifying the proof of the corresponding theorem in K. Kunugui's paper:

Theorem II. (The second property of Beurling-Kunugui) *Suppose that* $\Omega = S_{z_0}^{(D)} - S_{z_0}^{(\sigma)}$ *is non-empty, in addition to the assumptions of Theorem I, and denote by* Ω_n *and connected component of* Ω. *Then* $R_{z_0}^{(D)}$ *includes every value, with two possible exceptions, belonging to* Ω_n.

10) K. Kunugui, *Sur un problème de M. A. Beurling*, Proc. Imp. Acad. Tokyo, **16** (1940), pp. 361-366.