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1. Introduction

Frostman's theory\(^1\) on equilibrium potentials of order \(\alpha\) has been recently extended by Kunugui\(^2\) to generalized potentials. The purpose of this paper is to study the same problem from another point of view.

For preparation we state some definitions on generalized potentials and capacities. Denote by \(\mathcal{E}\) the whole Euclidean space, by \(\delta(E)\) the diameter of a bounded Borel set \(E\), by \(r_{r_0}\) the length of a segment \(PQ\), and by \(D_E^m\) the family of non-negative mass distributions of total mass \(m\) on a bounded Borel set \(E\); especially when \(m = 1\), we denote it by \(D_E\) simply. Let \(\Phi(t)\) be a strictly monotone decreasing and continuous function defined in the interval \((0, \infty)\) such that \(\lim_{t \to 0^+} \Phi(t) = +\infty\). Given any mass distribution \(\mu\) on \(E\), we call the Lebesgue-Stieltjes integrals

\[
\int_E \Phi(r_{r_0}) d\mu(Q) \quad \text{and} \quad \int_E \Phi(r_{r_0}) d\mu(Q) d\mu(P)
\]

the \(\Phi\)-potentials and the \(\Phi\)-energy integrals respectively with respect to \(\mu\). Put

\[
V_E^\Phi = \inf_{\mu \in D_E} \sup_{t \in (0, \infty)} \int_E \Phi(r_{r_0}) d\mu(Q) \quad \text{and} \quad W_E^\Phi = \inf_{\mu \in D_E} \int_E \Phi(r_{r_0}) d\mu(Q) d\mu(P),
\]

then it is easily seen that

\[
\Phi[\delta(E)] \leq V_E^\Phi \leq +\infty \quad \text{and} \quad \Phi[\delta(E)] \leq W_E^\Phi \leq +\infty.
\]

We define the \(\Phi\)-capacity \(C^\Phi(E)\) of \(E\) as follows; if \(V_E^\Phi < +\infty\), then \(C^\Phi(E) = \Phi^{-1}[V_E^\Phi]\), and if \(V_E^\Phi = +\infty\), then \(C^\Phi(E) = 0\), where \(\Phi^{-1}\) denotes the inverse function of \(\Phi\). Hereafter we shall write for the sake of simplicity \(V_E, W_E, C(E)\) for \(V_E^\Phi, W_E^\Phi, C^\Phi(E)\).

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2. Equilibrium potentials.

Let \( \Omega \) be the ordinary space and \( \phi(t) \) satisfy the following conditions:

a) \( \phi(t) \) is strictly monotone decreasing and continuous in \((0, \infty)\)
   and \( \lim_{t \to 0^+} \phi(t) = +\infty \).

b) \( t\phi(t) \) is convex in \((0, \infty)\). (What is the same, \( \phi(t) \) is convex
   in \( \frac{1}{t} \) in \((0, \infty)\)).

\[
\gamma \quad \int_0^\infty t^2\phi(t)dt < +\infty \quad \text{and} \quad \lim_{r \to 0^+} \int_0^r r^2\phi(t)dt < +\infty.
\]

Then, from \( \beta \), we see that \( \phi(r_{Q}) \) is subharmonic in \( \Omega - \{ Q \} \) when
\( Q \) is fixed in \( \Omega \). Accordingly, for any non-negative mass distribution \( \mu \) on a bounded closed set \( F \), the potential with respect to \( \mu \) is
subharmonic in each component of \( \Omega - F \).

First, let us consider the maximum principle: Let \( \phi(t) \) satisfy
the conditions \( a \) and \( \beta \) and \( f(P) \) be continuous and superharmonic
in \( \Omega \). If the potential \( u(P) \) with respect to a non-negative mass
distribution \( \mu \) whose kernel is a bounded closed set \( F \) is \( \leq f(P) \) in
\( F \), then \( u(P) \leq f(P) \) in \( \Omega \). In case \( \phi(t) = \frac{1}{t} \), it has been obtained
by Yosida\(^3\). His proof is very elementary and interesting. The
general case may be treated by a slight modification. Let us state
the proof briefly. For any \( \varepsilon > 0 \), take a closed subset \( F' \) of \( F \) such
that \( u(P) \) is continuous in \( F' \) and \( \mu(F - F') < \varepsilon \). Then
\[
\int_{E \subset (r', \delta')} \phi(r_{Q})d\mu(Q) < \varepsilon \quad \text{in} \quad F',
\]
where \( s(P, \delta) \) denotes any sphere with center \( P \in F' \) and radius \( \delta \) (a constant).
Therefore,
\[
\int_{E \subset (r', \delta')} \phi(r_{Q})d\mu(Q) < \varepsilon \quad \text{in} \quad F'.
\]
Accordingly, \( \int_{E} \phi(r_{Q})d\mu(Q) \) is continuous
in \( F' \); after all it become continuous in \( \Omega \).\(^5\)

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\(^3\) If \( \Omega \) is the plane, the conditions \( \beta \) and \( \gamma \) for \( \phi(t) \) must be replaced by the
following:

\( \beta' \) \( \phi(t) \) is convex in \( \log \frac{1}{t} \) in \((0, \infty)\).

\( \gamma' \) \( \int_0^\infty t^2\phi(t)dt < +\infty \) \( \text{and} \) \( \lim_{r \to 0^+} \int_0^r r^2\phi(t)dt < +\infty \).


Let $P^*$ be a point where $\int_{P'} \Phi(r_{r_{P_0}})d\mu(Q) - f(P)$ attains its maximum. Then $P^* \in F'$. Using this fact, for any point $P \in \Omega - F$, we easily see that

$$u(P) - f(P) \leq (\Phi(l) - \Phi(\delta(F)] \cdot \mu(F - F'),$$

where $l$ is the distance between $P$ and $F$. Thus we obtain $u(P) \leq f(P)$ in $\Omega - F$.

Using the maximum principle, we obtain at once the following two theorems.

**Theorem A:** Let $\Phi(t)$ satisfy the conditions $\alpha$) and $\beta$) and $F'$ be a bounded closed set of positive $\Phi$-capacity. Then there exists $\mu_0 \in D_F$ such that the $\Phi$-potential $u_0(P)$ with respect to $\mu_0$ is constant and equal to its maximum in $F$ except a possible set of $\Phi$-capacity 0.

**Theorem B:** Let $\Phi(t)$ and $F$ be the same as above and $f(P)$ be continuous and superharmonic in $\Omega$. Then there exists $\mu_0 \in D_F$ such that the $\Phi$-potential $u_0(P)$ with respect to $\mu_0$ is $f(P) + \gamma$ in $F$ except a possible set of $\Phi$-capacity 0 and always $\leq f(P) + \gamma$ in $\Omega$, where $\gamma$ is a suitable constant.

**Remark:** The mass distribution $\mu_0$ in Theorem A is what minimizes the energy integrals

$$I(\mu) = \int \int_{P'} \Phi(r_{r_{P_0}})d\mu(Q)d\mu(P)$$

with $\mu \in D_F$; while, the distribution $\mu_0$ in Theorem B is what minimizes Gauss variations

$$G(\mu) = \int \int_{P'} \Phi(r_{r_{P_0}})d\mu(Q)d\mu(P) - 2\int_{F'} f(P)d\mu(P)$$

with $\mu \in D_F$.

**Remark:** The following fact is very important: $W_E$ coincides with $V_E$ for any bounded Borel set $E$. It is easily proved from Theorem A and the properties of $W$ and $V$.

3. **Poincaré’s condition.**

Given a bounded Borel set $E$ and its limiting point $P_0$, we say that $P_0$ satisfies Poincaré's condition with respect to $E$, if there exists a cone such that its vertex is $P_0$ and its interior is contained in $E$. When $E$ is a bounded open set and $P_0$ is any point in $E$, then $P_0$ is a limiting point of $E$.

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6) O. Frostman: loc. cit. p. 56.
8) O. Frostman loc. cit. pp. 49–52.
in $E$. Now, we shall replace Poincaré's condition by another potential-theoretic condition.

We obtain:

**Theorem:** Let $\Phi(t)$ satisfy the conditions $a)$ and $\gamma$), and let $E$ any bounded Borel set, and $P_0$ its limiting point. If $\lim_{s \to r_0} m(E_s) > 0$, then $\lim_{s \to r_0} V_{E_s} > 0$, where $s$ denotes a sequence of closed spheres with center $P_0$.

**Proof:** There exist $\delta_0$ and $m$ such that $\Phi(t) > 0$ and
\[
\int_0^r \frac{t^2 \Phi(t)}{r^3} dt < m \quad \text{in} \quad (0, \delta_0).
\]
For any sphere $s$ with center $P_0$, it is easily seen that $\int_s \Phi(r_0) d\tau_Q$ is continuous in $\mathcal{O}$ and attains its maximum at $P_0$, where $d\tau_Q$ denotes a volume element at $Q$. Take a small sphere $s$ with center $P_0$ and radius $r < \frac{1}{6} \delta_0$. Then
\[
V_s \geq W_s = \inf_{\mu \in P_s} \int_s \Phi(r_0) d\mu(Q) d\mu(P) \geq \Phi(2r) = 4\pi \cdot (2r)^3 \cdot \frac{1}{24} \cdot \frac{1}{m(s)}
\]
and
\[
V_{E_s} = \inf_{\mu \in P_{E_s}} \sup_{Q \in \Omega} \int_{E_s} \Phi(r_0) d\mu(Q) \leq \frac{1}{m(E_s)} \sup_{Q \in \Omega} \int_{E_s} \Phi(r_0) d\tau_Q.
\]
Let $P^*$ be a point where $\int_{E_s} \Phi(r_0) d\tau_Q$, being continuous in $\mathcal{O}$, attains its maximum. Then we easily see that $r_{r_0,P^*} \leq 3r$.

Hence
\[
\int_{E_s} \Phi(r_0) d\tau_Q \leq \int_s \Phi(r_0) d\tau_Q \leq \int_s \Phi(r_0) d\tau_Q = 4\pi \int_0^{2r} t^2 \Phi(t) dt < 4\pi \int_0^{2r} t^2 \Phi(t) dt.
\]
Therefore
\[
V_{E_s} < \frac{4\pi}{m(E_s)} \int_0^{2r} t^2 \Phi(t) dt.
\]
Accordingly
\[
\frac{V_s}{V_{E_s}} > \frac{1}{24} \frac{(2r)^3 \cdot \Phi(2r) \cdot m(E_s)}{m(s)} > \frac{1}{24} \frac{m(E_s)}{m(s)}.
\]
Thus we obtain
\[ \lim_{s \to r_0} \frac{V_s}{V_{Es}} \geq \frac{1}{24m} \lim_{s \to r_0} \frac{m(Es)}{m(s)} > 0. \]

Hereafter we shall say that \( P_0 \) satisfies generalized \( \phi \)-Poincaré’s condition with respect to \( E \) if \( \lim_{s \to r_0} \frac{V_s}{V_{Es}} > 0 \).

**Theorem:** Let \( \phi(t) \) be the same as above, \( u(P) \) a potential with respect to a non-negative mass distribution \( \mu \) on a bounded closed set \( F \), \( E \) any bounded Borel set, and \( P_0 \) its limiting point. If \( P_0 \) satisfies generalized \( \phi \)-Poincaré’s condition with respect to \( E \), then holds
\[ u(P_0) = \lim_{E \to F \to P_0} u(P). \]

**Proof:** For any \( \lambda \leq 1 \) and \( r \leq \delta_0 \), we obtain \( \phi(\lambda r) \leq \frac{3m}{\lambda^3} \phi(r) \), since
\[ m \cdot r^3 \phi(r) > \int_0^{\lambda r} \frac{d\theta}{t^2} \phi(t) dt > \lambda^3 \int_0^{\lambda r} \frac{d\theta}{t^2} \phi(t) dt = \frac{\lambda^3 r^3}{3} \phi(\lambda r). \]

Therefore, for any \( \varepsilon < \min(\phi^{-1}(m), 1) \), we get
\[ \phi(\varepsilon r) < \frac{3m}{\varepsilon^3} \phi(r) < \frac{3}{\varepsilon^3} \phi(\varepsilon) \cdot \phi(r). \]

We have only to prove our theorem in the case when \( P_0 \in F_0 \) and \( u(P_0) < +\infty \), where \( F_0 \) denotes the kernel of \( \mu \). We can take a sphere \( S_0 \) with center \( P_0 \) and radius \( R_0 < \delta_0 \) such that
\[ \int_{S_0} \phi(r_{s_0}) d\mu(Q) < \varepsilon^6. \]

Put \( u_0(P) = \int_{S_0} \phi(r_{s_0}) d\mu(Q) \) and \( u_0(P) = \int_{F - S_0} \phi(r_{s_0}) d\mu(Q) \).

Take a concentric sphere \( s_0 \subseteq S_0 \) such that \( u_0(P) < u_0(P_0) + \varepsilon \) in \( s_0 \).

Let \( \lim_{r \to r_0} \frac{V_s}{V_{Es}} > K > 0 \). Then there exists a sequence of concentric spheres \( \{s_n\} \) such that \( s_0 > s_1 > s_2 > \cdots \to P_0 \) and \( \frac{V_{s_n}}{V_{Es_n}} > K > 0 \).

Take any \( s_n \) with radius \( r_n < \varepsilon R_0 \) and its concentric sphere \( S_n \) with radius \( R_n \left(= \frac{r_n}{\varepsilon} \right) \), then \( s_n \subseteq S_n \subseteq S_0 \) and \( \frac{\phi(r_n)}{\phi(R_n)} < \frac{3}{\varepsilon^3} \phi(\varepsilon) \).

As \( C(Es_n) > 0 \), there exists \( \mu_n \in D_{Es_n} \) such that
\[ \sup_{r \in s_n} \int_{Es_n} \phi(r_{s_0}) d\mu_n(Q) < 2V_{Es_n}. \]
We shall now evaluate the last two terms of the above expression. The second term is

\[
<2 V_{E_{n}} \cdot \mu(S_{a}) < \frac{2}{K} V_{S_{n}} \cdot \mu(S_{a})
\]

\[
< \frac{2}{K} \cdot \epsilon^{6} \cdot \sup_{P \in E_{n}} \frac{1}{4 \pi r_{a}^{3}} \int_{s_{a}} \phi(r_{P}) d \tau_{Q}
\]

\[
= \frac{6}{K} \cdot \epsilon^{6} \cdot \frac{1}{4 \pi r_{a}^{3}} \int_{s_{n}} \phi(r_{P}) d \tau_{Q}
\]

\[
= \frac{1}{K} \cdot \epsilon^{6} \cdot \frac{\int_{0}^{\epsilon} \phi(t) dt}{r_{a}^{3}} < \frac{6}{K} \cdot \epsilon^{6} \cdot m \phi(r_{n})
\]

\[
< \frac{6m}{K} \cdot \epsilon^{6} \cdot \frac{3}{\epsilon^{3}} \phi(\epsilon) = \frac{18m}{K} \cdot \epsilon^{3} \phi(\epsilon).
\]

Next, for any \(P \in s_{n}\) and any \(Q \in S_{0} - S_{a}\),

\[
\frac{r_{PQ}}{r_{P}} = 1 + \frac{r_{PQ} - r_{P}}{r_{P}} \leq 1 + \frac{r_{PQ} - r_{P}}{r_{a}} \leq 1 + \frac{r_{a}}{R_{a} - r_{a}}
\]

\[
= \frac{R_{a}}{R_{a} - r_{a}} = \frac{1}{1 - \epsilon}.
\]

Hence,

\[
r_{PQ} \geq (1 - \epsilon) r_{a}, \quad \phi(r_{P}) \leq \phi[(1 - \epsilon) r_{a}] \leq \frac{3m}{(1 - \epsilon)^{3}} \phi(r_{a}),
\]

and

\[
\int_{E_{n}} \phi(r_{P}) d \mu_{n}(P) \leq \frac{3m}{(1 - \epsilon)^{3}} \phi(r_{a}).
\]

Therefore, the first term of the above expression is

\[
\leq \frac{3m}{(1 - \epsilon)^{3}} \int_{s_{a} - s_{n}} \phi(r_{P}) d \mu(Q) \leq \frac{3m}{(1 - \epsilon)^{3}} \epsilon^{6}.
\]
Thus
\[ \int_{E_n} u(P) d\mu_n(P) = u(P_0) + A(\varepsilon), \]
where
\[ A(\varepsilon) = \varepsilon + \frac{3m}{(1-\varepsilon)^3} \varepsilon^6 + \frac{18m}{K} \varepsilon^2 \phi(\varepsilon). \]

Therefore, there exists a sequence of points \( \{P_n\} \) such that \( E \ni P_n \rightarrow P_0 \) and \( u(P_n) \leq u(P_0) + A(\varepsilon) \). Accordingly,
\[ \lim_{E \ni P_n \rightarrow P_0} u(P) \leq u(P_0) + A(\varepsilon). \]
Making \( \varepsilon \rightarrow 0 \), we obtain
\[ \lim_{E \ni P \rightarrow P_0} u(P) = u(P_0). \]
We have \( \lim_{E \ni P \rightarrow P_0} u(P) = u(P_0) \) from the lower semi-continuity of \( u(P) \).

**Theorem:** Let \( \Phi(t) \) satisfy the conditions \( \alpha \), \( \beta \), and \( \gamma \). Then the equilibrium potential \( u_0(P) \) in a bounded closed set \( F \) of positive \( \Phi \)-capacity attains its maximum \( V_\gamma \) at any point, of \( F \), which satisfies generalized \( \Phi \)-Poincaré's condition with respect to \( F \).

**Proof:** Put
\[ F^\prime = \{ P; P \in F, u_0(P) = V_\gamma \}. \]
Then
\[ C(F - F^\prime) = 0. \]
If \( \lim_{s \rightarrow P^*} \frac{V_s}{V_{P^*}} > 0 \), then \( P^* \) is a limiting point of \( F^\prime \) and \( \lim_{s \rightarrow P^*} \frac{V_s}{V_{P^*}} > 0. \)
Accordingly,
\[ u_0(P^*) = \lim_{F \ni P \rightarrow P^*} u(P) = V_\gamma. \]

**Remark:** Frostman has defined\(^9\) the capacity density of \( E \) at \( P_0 \) by \( \lim_{s \rightarrow P_0} \frac{C^\alpha(E_s)}{C^\alpha(s)} \) in his theory of potentials of order \( \alpha \). For a sufficiently small sphere \( s \),
\[ V_{E_s} = \phi \left[ C(E_s) \right] = \phi \left[ \frac{C(E_s)}{C(s)} \cdot C(s) \right] \leq \left[ \frac{3m}{C(E_s)} \right]^3 \phi \left[ \frac{C(s)}{C(E_s)} \right]^3 \cdot V_s. \]

Hence we obtain
\[ \lim_{s \rightarrow P_0} \frac{V_s}{V_{E_s}} \geq \frac{1}{3m} \left[ \lim_{s \rightarrow P_0} \frac{C(E_s)}{C(s)} \right]^3. \]

\(^9\) O. Frostman: loc. cit. p. 57.
Thus our generalized \( \phi \)-Poincaré's condition contains the density of \( \phi \)-capacity by Frostman's definition.

4. Energy integrals.

Kunugui\(^{10}\) has obtained the following important theorem: Let

\[ \phi(t) \text{ be a monotone increasing and convex function in } (0, \infty) \text{ and } \lim_{t \to 0} \phi(t) = 0. \]

Then for any completely additive function \( \sigma \) of Borel sets on a bounded closed set \( F \) in the ordinary space, an energy integral

\[
I(\sigma) = \int \int \phi\left(\frac{1}{r_{pq}}\right) d\sigma(Q) d\sigma(P)
\]

with respect to \( \sigma \), if it exists, is always \( \geq 0 \); especially the equality holds if and only if \( \sigma \equiv 0 \). In his proof he has used Fourier transformation of \( \phi\left(\frac{1}{r_{pq}}\right) \). Here, we will give another proof by using Theorem A and B. We suppose that \( \phi(t) \) satisfies the conditions \( \alpha \) and \( \beta \). For our proof we shall use the following six lemmas.

**Lemma 1**: Let \( F_1 \) and \( F_2 \) be two disjoint bounded closed sets of positive \( \Phi \)-capacity, \( \mu \) and \( \nu \) be two non-negative mass distributions of total mass unity whose kernels are \( F_1 \) and \( F_2 \) respectively. Then it is impossible that

\[
\gamma_1 = V_{F_1} - \int_{F_2} d\nu(P) \int_{F_1} \phi(r_{pq}) d\mu_0(Q)
\]

and

\[
\gamma_2 = V_{F_2} - \int_{F_1} d\mu(P) \int_{F_2} \phi(r_{pq}) d\nu_0(Q)
\]

vanish simultaneously, where \( \mu_0 \) and \( \nu_0 \) denote equilibrium distributions on \( F_1 \) and \( F_2 \) respectively.

**Proof**: Evidently \( \gamma_1 \geq 0 \) and \( \gamma_2 \geq 0 \). We easily see that the equilibrium potential \( u_0(P) \) on a bounded closed set \( F \) of positive \( \Phi \)-capacity is \( < V_{\nu} \) in \( \Omega^\nu \), where \( \Omega^\nu \) denotes the component of \( \Omega - F \) which contains the infinity. Either \( F_1 \cdot \Omega^\nu_2 \) or \( F_2 \cdot \Omega^\nu_1 \) is not empty. Suppose that \( F_1 \cdot \Omega^\nu_2 \) is not empty and \( P_0 \) is its arbitrary point. Then \( V_{F_2} > \int_{F_2} \phi(r_{pq}) d\nu_0(Q) \). Consequently, \( V_{F_2} > \int_{F_2} \phi(r_{pq}) d\nu_0(Q) \) in some neighbourhood \( U(P_0) \) of \( P_0 \).

\(^{10}\) K. Kunugui: loc. cit. (II).
Being
\[ \mu(U(P_0)) > 0, \quad V_{x_2} \cdot \mu(U(P_0)) \geq \int_{\alpha r_2} d\mu(p) \int_{\alpha r_2} \Phi(r_{r_2}) d\nu_0(Q). \]

But certainly
\[ V_{x_2} \cdot [\mu(F_r) - \mu(U(P_0))] \geq \int_{F_r - U(P_0)} d\mu(P) \int_{F_r} \Phi(r_{r_2}) d\nu_0(Q). \]

Thus,
\[ V_{x_2} \geq \int_{F_r} d\mu(P) \int_{F_r} \Phi(r_{r_2}) d\nu_0(Q). \]

**Lemma 2:** Let \( E \) be any bounded Borel sets and \( \mu, \nu \in D_E \) such that
\[ \int\int_E \Phi(r_{r_2}) d\mu(Q) d\mu(P) < +\infty \]
and
\[ \int\int_E \Phi(r_{r_2}) d\nu(Q) d\nu(P) < +\infty. \]

If
\[ \int\int_E \Phi(r_{r_2}) d\mu(Q) - \int\int_E \Phi(r_{r_2}) d\nu(Q) \]
\( = \varepsilon^{11} \) (a constant) in \( E \), then \( \mu = \nu. \)

**Proof:** Let \( \sigma = \mu - \nu \) and suppose \( \sigma \equiv 0 \). Let \( \sigma = \mu' - \nu' \) be Hahn's decomposition of \( \sigma \). Then there exists a closed sphere \( s^{12} \) such that \( \sigma(s) = \alpha > 0 \), and we can take its concentric closed sphere \( S \) such that \( S \supset s \) and \( \nu'(S - s) \leq \varepsilon < \alpha \). Let \( f_1(P) = \int_{\phi_{r_{r_2}} s} \frac{1}{r_{r_2}} d\sigma_m \), \( s^o \) denoting a surface of \( s \) and \( d\sigma_m \) a surface element at \( M \in s^o \). Then,
\[ f_1(P) \equiv A \text{ in } s \text{ and } \equiv B \text{ on } S^o, \]
where both \( A \) and \( B \) are constants and \( A > B > 0 \). \( f_1(P) \) is continuous and superharmonic in \( \Omega \). Let \( f_2(P) \equiv B \) in \( S \) and \( \equiv f_1(P) \) in \( \Omega - S \). Then \( f_2(P) \) is also continuous and superharmonic in \( \Omega \). We see \( C(E) > 0 \) from
\[ \int\int_E \Phi(r_{r_2}) d\mu(Q) d\mu(P) < \infty \]
and
\[ \int\int_E \Phi(r_{r_2}) d\nu(Q) d\nu(P) < +\infty. \]

By Theorem B, there exist \( \mu_1, \mu_2 \in D_E \) such that

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11) \( \varepsilon \) means the coincidence except a possible set of \( \phi \)-capacity 0.
12) O. Frostman: loc. cit. p. 32.
\[
\int_E \varphi(r_{r_0}) d\mu_1(Q) \equiv f_1(P) + \gamma_1
\]

and
\[
\int_E \varphi(r_{r_0}) d\mu_2(Q) \equiv f_2(P) + \gamma_2
\]
in \( \bar{E} \), where \( \gamma_1 \) and \( \gamma_2 \) are suitable constants. Put
\[
f(P) = \int_E \varphi(r_{r_0}) d\mu_1(Q) - \int_E \varphi(r_{r_0}) d\mu_2(Q).
\]

Then \( f(P) \equiv f_1(P) - f_2(P) + \gamma \) in \( \bar{E} \), where \( \gamma = \gamma_1 - \gamma_2 \). Clearly, \( f(P) \equiv A - B + \gamma \) in \( \bar{E} \cdot \delta \), \( \gamma \leq f(P) \leq A - B + \gamma \) in \( \bar{E} \cdot (S - \delta) \) except a possible set of \( \Phi \)-capacity 0, and \( \equiv \gamma \) in \( \bar{E} - S \). Next consider
\[
\int_{\bar{E}} f(P) d\sigma(P) = \int_{\bar{E}} f(P) d\sigma(P) = \int_{\bar{E} \cdot \delta} + \int_{\bar{E} \cdot (S - \delta)} + \int_{\bar{E} - S} f(P) d\sigma(P).
\]
Of course \( \sigma \) cannot have any mass on a set of \( \Phi \)-capacity 0, since
\[
\iint_E \varphi(r_{r_0}) d\mu(Q) d\mu(P) < + \infty
\]
and
\[
\iint_E \varphi(r_{r_0}) d\nu(Q) d\nu(P) < + \infty.
\]
\[
\int_{\bar{E} \cdot \delta} f(P) d\sigma(P) = (A - B + \gamma) \cdot \sigma(\delta), \quad \int_{\bar{E} \cdot (S - \delta)} f(P) d\sigma(P) = \gamma \cdot \sigma(\bar{E} - S),
\]
\[
\int_{\bar{E} - S} f(P) d\sigma(P) = \int_{\bar{E} \cdot (S - \delta)} f(P) d\mu'(P) - \int_{\bar{E} \cdot (S - \delta)} f'(P) d\nu'(P)
\]
\[
\geq \gamma \cdot \mu'(S - \delta) - (A - B + \gamma) \cdot \nu'(S - \delta) > -(A - B) \cdot \varepsilon + \gamma \cdot \sigma(S - \delta)
\]
Therefore,
\[
\int_{\bar{E}} f(P) d\sigma(P) > (A - B)(\alpha - \varepsilon) + \gamma \cdot \sigma(\bar{E}) > \gamma \cdot \sigma(\bar{E}) = 0.
\]
We have however, denoting \( \kappa = \mu_1 - \mu_2 \),
\[
\int_{\bar{E}} f(P) d\sigma(P) = \int_{\bar{E}} d\sigma(P) \int_{\bar{E}} \phi(r_{r_0}) d\kappa(Q)
\]
\[
= \int_{\bar{E}} d\kappa(Q) \int_{\bar{E}} \phi(r_{r_0}) d\sigma(P) = 0,
\]
which is a contradiction.

**Corollary:** The equilibrium distribution on a bounded closed set of positive \( \Phi \)-capacity is unique.
Hereafter, we suppose \( \Phi(t) \geq 0 \) in \((0, \infty)\). When \( E_1 \) and \( E_2 \) are disjoint bounded Borel sets of positive \( \Phi \)-capacity, for \( \mu \in D_{E_1} \) and \( \nu \in D_{E_2} \) we put

\[
G [\mu \in D_{E_1}, \nu \in D_{E_2}] = \frac{\iint_{E_1} \Phi(r_{ro}) d\mu(Q) d\mu(P) \times \iint_{E_2} \Phi(r_{ro}) d\nu(Q) d\nu(P)}{\iint_{E_1} d\mu(P) \iint_{E_2} \Phi(r_{ro}) d\nu(Q)}.
\]

So far as there is no confusion, we shall write \( G(\mu, \nu) \) for \( G [\mu \in D_{E_1}, \nu \in D_{E_2}] \).

**Lemma 3:** Let \( E_1 \) and \( E_2 \) be two sets stated above. If a pair \((\mu^*, \nu^*)\), \( \mu^* \in D_{E_1} \) and \( \nu^* \in D_{E_2} \), minimizes \( G(\mu, \nu) \), then \((\mu^*, \nu^*)\) has the following properties;

i) \( C \{ P; P \in E_1, g_1(P) < 0 \} = 0 \) and \( \mu^* \{ P; P \in E_1, g_1(P) > 0 \} = 0 \),

ii) \( C \{ P; P \in E_2, g_2(P) < 0 \} = 0 \) and \( \nu^* \{ P; P \in E_2, g_2(P) > 0 \} = 0 \),

where

\[
x = \iint_{E_1} \Phi(r_{ro}) d\mu^*(Q) d\mu^*(Q),
\]

\[
y = \iint_{E_2} \Phi(r_{ro}) d\nu^*(Q) d\nu^*(P),
\]

\[
z = \int_{E_1} d\mu^*(P) \int_{E_2} \Phi(r_{ro}) d\nu^*(Q),
\]

\[
g_1(P) = 2 \int_{E_1} \Phi(r_{ro}) d\mu^*(Q) - 2x \int_{E_2} \Phi(r_{ro}) d\nu^*(Q)
\]

and

\[
g_2(P) = 2 \int_{E_2} \Phi(r_{ro}) d\nu^*(Q) - 2y \int_{E_1} \Phi(r_{ro}) d\mu^*(Q)
\]

under the assumption \( 0 < x, y, z < + \infty \).

**Proof:** For any \( \delta > 0 \), put

\[
A = \{ P; P \in E_1, g_1(P) > -\delta \} \quad \text{and} \quad B = \{ P; P \in E_1, g_1(P) < -2\delta \}.
\]

Then \( A \cdot B = 0 \) and \( \mu^*(A) > 0 \) from \( \int_{E_1} g_1(P) d\mu^*(P) = 0 \). Suppose \( C(B) > 0 \). Then we can distribute a new mass \( \tau \) on \( E_1 \) such that \( \tau = -\mu^* \) on \( A \), \( \tau \geq 0 \) on \( B \) and \( \tau(B) = \mu^*(A) = a > 0 \), \( \tau = 0 \) on \( E_1 - (A + B) \) and \( \sup P \tau \int_{E_1} \Phi(r_{ro}) d\tau(Q) < + \infty \). As \( \mu^* + \varepsilon \tau \in D_{E_1} \) for any positive number \( \varepsilon < 1 \), \( G(\mu^* + \varepsilon \tau, \nu^*) \geq G(\mu^*, \nu^*) \). Hence,
We can easily see that the coefficient of $\varepsilon^2$ is finite, and so the last side becomes $<0$ for some positive number $\varepsilon < 1$, which is a contradiction. Therefore, $C(B) = 0$. Making $\delta \to 0$, we obtain

$$C\{P; P \in E_1, g_1(P) < 0\} = 0.$$ 

Accordingly, $\mu^*\{P; P \in E_1, g_1(P) < 0\} = 0$,

and since

$$\iint_{E_1} \phi(r_{r_0}) d\mu^*(Q) d\mu^*(P) < +\infty.$$ 

Thus, $\mu^*\{P; P \in E_1, g_1(P) > 0\} = 0$.

Similarly, we obtain

$$C\{P; P \in E_2, g_2(P) < 0\} = 0$$

and

$$\nu^*\{P; P \in E_2, g_2(P) > 0\} = 0.$$ 

Lemma 4: Let $F_1$ and $F_2$ be two disjoint closed sets of positive $\phi$-capacity, then $G(\mu, \nu) > 1$ for any $\mu \in D_{F_1}$ and $\nu \in D_{F_2}$.

Proof: Let $G^* = \inf_{\mu \in D_{F_1}, \nu \in D_{F_2}} G(\mu, \nu)$. Then there exist $\{\mu_i\} \in D_{F_1}$ and $\{\nu_i\} \in D_{F_2}$ such that $G(\mu_i, \nu_i) \downarrow G^*$. We may suppose that both $\{\mu_i\}$ and $\{\nu_i\}$ are convergent. Let $\mu^* \in D_{F_1}$ and $\nu^* \in D_{F_2}$ be respectively their limiting mass distributions. Then $G^* \leq G(\mu^*, \nu^*)$. On the other hand,

$$\iint_{F_1} \phi(r_{r_0}) d\mu^*(Q) d\mu^*(P) \leq \lim_{i \to \infty} \iint_{F_1} \phi(r_{r_0}) d\mu_i(Q) d\mu_i(P),$$

$$\iint_{F_2} \phi(r_{r_0}) d\nu^*(Q) d\nu^*(P) \leq \lim_{i \to \infty} \iint_{F_2} \phi(r_{r_0}) d\nu_i(Q) d\nu_i(P)$$

and

$$\int_{F_1} d\mu^*(P) \int_{F_2} \phi(r_{r_0}) d\nu^*(Q) = \lim_{i \to \infty} \int_{F_1} d\mu_i(P) \int_{F_2} \phi(r_{r_0}) d\nu_i(Q),$$

because $\phi(r_{r_0})$ is bounded and continuous for $P \in F_1$ and $Q \in F_2$.

Hence, $G(\mu^*, \nu^*) \leq \lim_{i \to \infty} G(\mu_i, \nu_i) = G^*$.

Therefore, $G^* = G(\mu^*, \nu^*)$. For $\mu^*$ and $\nu^*$, let $x, y, z, g_1(P)$ and
Let \( g_0(P) \) be as was stated in the previous lemma. Then evidently
\( 0 < x, y, z < +\infty \). Let
\[
F_1' = \{ P; P \in F_1, g_1(P) \leq 0 \}
\]
and \( F_1^* \) be the kernel of \( \mu^* \). We show that \( F_1^* \subset F_1' \). Suppose that there exists \( P_0 \in F_1^* \) such that \( g_1(P_0) > 0 \). Then \( g_1(P) > 0 \) in some neighbourhood \( U(P_0) \) of \( P_0 \) by the lower semi-continuity of \( g_1(P) \) at \( P_0 \). As \( \mu^*[U(P_0)] \geq 0 \), we obtain
\[
\mu^* \{ P; P \in F_1, g_1(P) > 0 \} \geq 0.
\]
which is a contradiction. Now, let \( \mu_0^* \) be the equilibrium distribution on \( F_1^* \) (of positive \( \Phi \)-capacity). Then
\[
0 \geq \int_{F_1^*} g_1(P) \, d\mu_0^*(P) = z^2 V_{F_1^*} - z x \int_{F_1^*} \, d\nu_0^*(Q) \int_{F_1^*} \Phi(r_{Q}) \, d\mu_0^*(P).
\]
Hence we see \( z^2 \leq z x \). Similarly, let \( \nu_0^* \) be the equilibrium distribution on the kernel \( F_2^* \) of \( \nu^* \), then
\[
0 \geq \int_{F_2^*} g_2(P) \, d\nu_0^*(P) = z^2 V_{F_2^*} - z y \int_{F_2^*} \, d\mu_0^*(Q) \int_{F_2^*} \Phi(r_{Q}) \, d\nu_0^*(P).
\]
Hence we see \( z^2 \leq z y \). It is impossible by Lemma 1 that \( z^2 = z x \) and \( z^2 = z y \) hold simultaneously. Thus, we obtain \( G^* = \frac{xy}{z^2} > 1 \).

**Lemma 5:** Let \( E_1 \) and \( E_2 \) be two disjoint bounded Borel sets of positive \( \Phi \)-capacity. Then \( G(\mu, \nu) \geq 1 \) for \( \mu \in D_{E_1} \) and \( \nu \in D_{E_2} \).

**Proof:** Let \( \{ F_n^{(1)} \} \) and \( \{ F_n^{(2)} \} \) be sequences of closed subsets of \( E_1 \) and \( E_2 \) respectively such that
\[
F_1^{(1)} \subset F_2^{(1)} \subset F_3^{(1)} \subset \cdots \subset E_1,
\]
\[
F_1^{(2)} \subset F_2^{(2)} \subset F_3^{(2)} \subset \cdots \subset E_2,
\]
\( \mu(F_n^{(1)}) \uparrow \mu(E_1) \) and \( \nu(F_n^{(2)}) \uparrow \nu(E_2) \); then we can easily see that \( C(F_n^{(1)}) \geq 0 \) and \( C(F_n^{(2)}) \geq 0 \) for sufficiently large \( n \). By Lemma 4,
\[
G_n(\mu, \nu) = \left( \frac{1}{\mu(F_n^{(1)})} \right)^2 \left( \frac{1}{\nu(F_n^{(2)})} \right) \left( \int_{F_n^{(1)}} \Phi(r_{Q}) \, d\mu(Q) \right) \times \left( \frac{1}{\nu(F_n^{(2)})} \right)^2 \left( \int_{F_n^{(2)}} \Phi(r_{Q}) \, d\nu(Q) \right) \geq 1.
\]
Making \( n \to \infty \), we obtain \( G(\mu, \nu) \geq 1 \).

**Lemma 6:** Under the same condition as in Lemma 5, \( G(\mu, \nu) > 1 \) for \( \mu \in D_{E_1} \) and \( \nu \in D_{E_2} \).

**Proof:** Suppose that \( G(\mu^*, \nu^*) = 1 \) for \( \mu^* \in D_{E_1} \) and \( \nu^* \in D_{E_2} \). Then \((\mu^*, \nu^*)\) minimizes \( G[\mu \in D_{E_1}, \nu \in D_{E_2}] \). For \( \mu^* \) and \( \nu^* \), let \( x, y, z, g_1(P) \) and \( g_2(P) \) be same as in Lemma 3, and

\[
E'_1 = \{ P; P \in E_1, g_1(P) \leq 0 \} \quad \text{and} \quad E''_1 = \{ P; P \in E_1, g_1(P) > 0 \};
\]

then evidently \( z^2 = xy \) and \( \sqrt{\frac{y}{x}} g_1(P) + \sqrt{\frac{x}{y}} g_2(P) = 0 \). As

\[
\mu^* \in D_{E'_1}, \nu^* \in D_{E_2} \subset D_{E'_1 + E''_1};
\]

by Lemma 3 and

\[
G(\mu^*, \nu^*) = 1, \ (\mu^*, \nu^*)
\]

also minimizes

\[
G[\mu \in D_{E'_1}, \nu \in D_{E_2} + E''_1]
\]

by Lemma 5. Hence, by Lemma 3

\[
C\{ P; P \in E_1 + E_2 + E''_1, g_2(P) < 0 \} = 0,
\]

and so

\[
C\{ P; P \in E_1 + E_2, g_2(P) < 0 \} = 0.
\]

Namely,

\[
C\{ P; P \in E_1 + E_2, \sqrt{\frac{y}{x}} g_1(P) > 0 \} = 0.
\]

Thus, \( C(E_1') = 0 \).

Accordingly,

\[
C\{ P; P \in E_1, g_1(P) = 0 \} = 0.
\]

Similarly, we see

\[
C\{ P; P \in E_2, g_2(P) = 0 \} = 0
\]

and so

\[
C\{ P; P \in E_2, \sqrt{\frac{y}{x}} g_1(P) = 0 \} = 0.
\]

Thus we obtain \( g_1(P) \leq 0 \) in \( E_1 + E_2 \). Next we shall show that \( g_1(P) \leq 0 \) in \( E_1 + E_2 \). Let \( M \) be any bounded Borel set contained in \( E_1 + E_2 \). As \( \mu^* \in D_{E_1}, \nu^* \in D_{E_2} + M \) and \( G(\mu^*, \nu^*) = 1, \ (\mu^*, \nu^*) \) also minimizes \( G[\mu \in D_{E_1}, \nu \in D_{E_2} + M] \) by Lemma 5. Hence,
and so \( C\{P; P \in M, g(P) < 0\} = 0 \).

Therefore, \( C\{P; P \in M, \sqrt{\frac{y}{x}} g(P) > 0\} = 0 \).

Similarly, \( C\{P; P \in E_1 + M, g_1(P) < 0\} = 0 \),
and so \( C\{P; P \in M, g_1(P) < 0\} = 0 \).

Thus, \( C\{P; P \in M, g_1(P) = 0\} = 0 \).

As \( M \) may be taken arbitrarily in \( \overline{E_1 + E_2} = (E_1 + E_2) \),
we obtain \( g_1(P) \equiv 0 \) in \( \overline{E_1 + E_2} \). Let \( \mu_0 \) be the equilibrium distribution on \( E_1 + E_2 \). Then
\[
0 = \int_{E_1 + E_2} g_1(P) d\mu_0(P) = \frac{1}{2} \int_{E_1 + E_2} x^2 V_{E_1 + E_2} - \frac{1}{2} \int_{E_1 + E_2} x^2 V_{E_1 + E_2}
\]
and so \( z = x \).

Finally, \( \int_{E_1} \phi(r_{r_0}) d\mu^*(Q) = \int_{E_2} \phi(r_{r_0}) d\nu^*(Q) \)
in \( \overline{E_1 + E_2} \), which contradicts to Lemma 2.

**Corollary:** Under the same condition
\[
\int_{E_1} \phi(r_{r_0}) d\mu(Q) d\mu(P) \times \int_{E_2} \phi(r_{r_0}) d\nu(Q) d\nu(P) > 1
\]
for \( \mu \in D_{E_1}^m \) and \( \nu \in D_{E_2}^n \), where \( m \) and \( n \) are arbitrary positive numbers.

As an immediate consequence of this corollary, we obtain

**Theorem:** Let \( \phi(t) \geq 0 \) and satisfy the conditions \( a) \) and \( \beta) \).

Then an energy integral
\[
I(\sigma) = \int_{E} \phi(r_{r_0}) d\sigma(Q) d\sigma(P)
\]
with respect to any mass distribution \( \sigma \) on a bounded Borel set \( E \), if it exists, is always \( \geq 0 \); especially the equality holds if and only if \( \sigma = 0 \).

**Remark:** In our proof, we can easily see that the non-negativity
of $\Phi(t)$ has only to be supposed in $(0, \delta(E))$. Accordingly, we obtain the following corollary: Let $\Phi(t)$ satisfy the conditions $\alpha$ and $\beta$. Then an energy integral

$$I(\sigma) = \iint_E \Phi(r_{pq}) d\sigma(Q) d\sigma(P)$$

with respect to any mass distribution $\sigma$ of algebraic 0 on a bounded Borel set $E$, if it exists, is always $\geq 0$; especially the equality holds if and only if $\sigma \equiv 0$. 