69. On the Behaviour of a Pseudo-Regular Function in a Neighbourhood of a Closed Set of Capacity Zero.

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Since R. Nevanlinna[1] established many important theorems on the behaviour of a meromorphic function in a neighbourhood of a closed set of capacity zero, many results on this problem have been obtained. M. Tsuji[2] has found that Evans' theorem plays an important role in such investigations and obtained many theorems systematically.

Pseudo-regular functions were first studied by H. Grötzsch[3]. He has proved that Picard's theorem can be generalized to a class of pseudo-regular functions and his theorem has been extended by M. Lavrentieff[4].

We have never known the study on the behaviour of a pseudo-regular function in a neighbourhood of a closed set of capacity zero, so that we treat with this problem. The object of this paper is to obtain generalizations of the theorems of Nevanlinna such as an extension of Liouville's theorem and that of the principle of maximum to a class of pseudo-regular functions.

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1.[5] A uniform continuous function

$$w = f(z) = u(x, y) + iv(x, y), \ z = x + iy$$

is called pseudo-regular in a domain, if it satisfies the following three conditions

(1) u_x, u_y, v_x, v_y exist and are continuous in the domain.

(2) $J(z) = u_x v_y - u_y v_x > 0$ except possibly at most the countable set of points which has no point of accumulation inside of the domain.

(3) At the point z_0 where $J(z_0) = 0$, a sufficiently small neighbourhood of z_0 is transformed topologically on a neighbourhood of an algebraic branch point at $w_0 = f(z_0)$ of a Riemann surface.

A transformation by a pseudo-regular function is called to be

1) R. Nevanlinna, Eindeutige analytische Funktionen (1936), pp. 130–136.
2) M. Tsuji, Jap. Jour. of Math. 19 (1944).
3) H. Grötzsch, Leipziger Berichte, 80 (1929).
4) M. Lavrentieff, C. R. 200 (1935).
5) S. Kakutani, Jap. Jour. of Math. 13 (1937), O. Teichmüller, Deutsche Math. 3 (1938.)

pseudo-conformal. A pseudo-conformal tranformation is topologically equivalent to a conformal representation.

At a point where $J(z)>0$, an infinitesimal circle with centre z is transformed by $w=f(z)$ into an infinitesimal ellipse with centre w, if we neglect infinitesimals of higher orders, and the magnitude of the ratio of major and minor axis of this ellipse is given by

$$D(z)=D_{z|w}=\mathrm{Max}\left|\frac{dw}{dz}\right|\Big/\mathrm{Min}\left|\frac{dw}{dz}\right|,$$

where

$$\frac{dw}{dz}=\frac{(u_x+iv_x)\cos\theta+(u_y+iv_y)\sin\theta}{\cos\theta+i\sin\theta}.$$

$D(z)$ is called the 'Dilatationsquotient' of $f(z)$ at the point z. We have

$$D_{z|w}=D_{w|z}$$

and

$$D(z)+\frac{1}{D(z)}=\frac{u_x^2+u_y^2+v_x^2+v_y^2}{u_xv_y-u_yv_x}.$$

If $w=f(z)$ is pseudo-regular and $\omega=g(w)$, $z=h(\zeta)$ are regular, then $\omega=g(f(h(\zeta)))$ is pseudo-regular and $D_{z|w}=D_{\zeta|\omega}$.

Between the areal elements of the $z(=x+iy)$-plane and the corresponding $w(=u+iv)$-plane there exists the relation

$$D(z)\left|\frac{dw}{dz}\right|^2dxdy\geqq dudv\geqq\frac{\left|\frac{dw}{dz}\right|^2}{D(z)}dxdy.$$

A point z_0 on the boundary is called a point of essential singularity of a pseudo-regular function $f(z)$, if there exists at least two sequences of points $\{z_n\}$ and $\{z_n'\}$ each of which tends to z_0 as $n\to\infty$ and for which the limits $\lim f(z_n)$ and $\lim f(_n')$ exist and are not equal to each other.

2. Let E be a bounded closed set of points on the z-plane. The potential

$$u(z)=\int_E\log\frac{1}{|z-a|}d\mu(a),\ \text{where}\ \int_E d\mu(a)=1$$

of a distribution of positive measure $d\mu(a)$ on E is harmonic in a domain bounded by E except at the infinity $z=\infty$. Let $v(z)$ be the conjugate harmonic function of $u(z)$ and put

$$t=\varphi(z)=u(z)+iv(z).$$

The function $\varphi(z)$ is many-valued regular outside E except at the infinity $z=\infty$, the infinity being a logarithmic singularity, and the multiformity of $\varphi(z)$ arises only in its imaginary part by some additive constants. Let C_u be the niveau curve: $u(z)=\text{const.}$

$=u$, then C_u consists of a finite number of Jordan curves and surrounds E if u is smaller than the minimum of $u(z)$ on E.

Let C be a Jordan curve or a finite number of Jordan curves surrounding E, then we have

$$\int_C dv(z) = \int_C \frac{\partial u}{\partial n} ds = 2\pi \int_E d\mu(a) = a\pi,$$

where ds is the arc length on C and n is the inner normal of C.

Theorem of Evans[6]. If E is of capacity zero, then there exists a distribution $d\mu(a)$ such that its potential $u(z)$ is positively infinite at every point of E and at no other points.

Theorem of Nevanlinna[7]. If E is of capacity positive, then the Robin's constant γ of E is finite and we can choose a distribution $d\mu(a)$ such that its potential $u(z)$ is not greater than γ at every points of E.

Theorem 1. Let D be a finite domain on the z-plane bounded by a Jordan curve C and a closed set E of capacity zero lying inside C and $\zeta=f(z)$ be the function which maps D one-to-one and pseudo-conformally on a finite domain G on the ζ-plane bounded by a Jordan curve L and a closed set F of capacity positive lying inside L such that C corresponds to L. Then the integral

$$\int^{\infty} \frac{du}{C(u)} \tag{1}$$

converges, where $C(u)$ is the maximum of the 'Dilatationsquotient' of $f(z)$ on the niveau curve C_u of the potential $u(z)$ of Evans' theorem.

Proof. Since E is of capacity zero, there exists the potential $u(z)$ of Evans' theorem on the z-plane. We may assume that C coincides with some niveau curve C_u. Let $v(z)$ be the conjugate harmonic function of $u(z)$ and put

$$t=\varphi(z)=u(z)+iv(z).$$

Since F is of capacity positive, there exists the potential $\omega(\zeta)$ of Nevanlinna's theorem on the ζ-plane. As G is a finite domain, $\omega(\zeta)$ is also lower bounded in G, so that $\gamma > \omega(\zeta) > \beta$, where β is a constant. Let $\sigma(\zeta)$ be the conjugate harmonic function of $\omega(\zeta)$ and pnt

$$\tau=\chi(\zeta)=\omega(\zeta)+i\sigma(\zeta).$$

Let L_u be the image of C_u by $\zeta=f(z)$, then L_u consists of a finite number of Jordan curves surrounding F, so that we have

$$\int_{L_u} d\sigma(\zeta) = 2\pi.$$

6) G. C. Evans, Monatsheft f. Math. u. Phys. 43 (1936).
7) R. Nevanlinna, loc. cit. pp. 122–130.

Since

$$\int_{C_u} dv(z) = 2\pi$$

and

$$\int_{L_u} d\sigma(\zeta) = I\int_{L_u} d\chi(\zeta) = I\int_{C_u} \frac{d\tau}{dt}\, d\varphi(z) = R\int_0^{2\pi} \frac{d\tau}{dt}\, dv,$$

we have by the inequality of Schwarz

$$(2\pi)^2 \leqq \int_0^{2\pi} dv \int_0^{2\pi} \left|\frac{d\tau}{dt}\right|^2 dv = 2\pi \int_0^{2\pi} \left|\frac{d\tau}{dt}\right|^2 dv\,.$$

Since the function $\varphi(z)$ and $\chi(\zeta)$ are regular, we have $D_{z|\tau} = D_{\zeta|\tau}$, so that we have

$$\frac{\left|\dfrac{d\tau}{dt}\right|^2}{C(u)}\, du dv \leqq d\omega d\sigma\,.$$

Hence

$$2\pi \int_{u_0}^{u} \frac{du}{C(u)} \leqq \int_{u_0}^{u} \left(\int_0^{2\pi} \frac{\left|\dfrac{d\tau}{dt}\right|^2}{C(u)}\, dv \right) du \leqq \int_{\beta}^{\gamma} \int_0^{2\pi} d\omega d\sigma = 2\pi(\gamma-\beta),$$

so that the integral (1) converges.

If E consists of only a point $z = a$, then D and G are duobly connected, $u(z) = -\log|z-a|$ and the niveau curve C_u is the circle $|z-a| = e^{-u}$. Put $r = e^{-u}$ and $C(u) = D(r)$, then we have the following theorem of Teichmüller[8] as the corollary of this theorem.

Corollary. If the domain $0 < |z-a| < 1$ is mapped by $\zeta = f(z)$ one-to-one and pseudo-conformally onto a finite domain bounded by a Jordan curve and a closed set containing at least two points, then the integral

$$\int_0^1 \frac{dr}{rD(r)}$$

converges, where $D(r)$ is the maximum of the 'Dilatationsquotient' of $f(z)$ on $|z-a| = r$.

Theorem 2. Let $\zeta = f(z)$ be the function which maps a simply connected finite domain D on the z-plane one-to-one and pseudo-conformally onto the unit-circle $|\zeta| < 1$. If a closed set E of capacity zero lying on the boundary of D corresponds to a closed set F of capacity positive lying on $|\zeta| = 1$ by this mapping, then the integral

$$\int^{\infty} \frac{du}{C'(u)} \tag{2}$$

8) O. Teichmüller, loc. cit.

converges, where $C'(u)$ is the maximum of the 'Dilatationsquotient' of $f(z)$ on the common part C'_u of D and the niveau curve C_u.

Proof. Since E is of capacity zero and F is of capacity positive, we have the same functions $\varphi(z)$ and $\chi(\zeta)$ as in the proof of Theorem 1 and $\omega(\zeta)$ is bounded: $\gamma > \omega(\zeta) > \beta$ in $|\zeta| < 1$.

If u_0 is a sufficiently large number, then C_u intersects the domain D for every $u \geqq u_0$. Let L'_u be the image of C'_u on the unit-circle $|\zeta| < 1$ and K_u be the part of the circle $|\zeta| = 1$ which corresponds to the boundary of D outside C_u, then we have

$$\int_{K_u + L'_u} d\sigma(\zeta) = 0$$

and

$$\int_{K_u} d\sigma(\zeta) = I\int_{K_u} d\left(\int_F \log\frac{1}{\zeta - a}\, d\mu(a)\right)$$

$$= -\frac{1}{2}\int_{K_u} darg\zeta \int_F d\mu(a) = -\frac{1}{2}\int_{K_u} darg\zeta = -\frac{1}{2}1(K_u),$$

where $1(K_u)$ is the length of K_u. Hence

$$\int_{L'_u} d\sigma(\zeta) = -\int_{K_u} d\sigma(\zeta) = \frac{1}{2}1(K_u).$$

Since $1\ (K_u)$ increases monotonically with u, we have

$$\int_{L'_u} d\sigma(\zeta) \geqq \int_{L'_{u_0}} d\sigma(\zeta) = 2\pi c$$

for $u \geqq u_0$, where c is a positive constant. Since

$$\int_{L'_u} d\sigma(\zeta) = R\int_{C'_u} \frac{d\tau}{dt}\, dv(z)$$

and

$$\int_{C'_u} dv(z) \leqq 2\pi,$$

we have similarly as in the proof of Theorem 1

$$2\pi c^2 \int_{v_0}^{u} \frac{du}{C'(u)} \leqq 2\pi(\gamma - \beta).$$

Hence the integral (2) converges.

Corollary. Let D be a domain and E be a closed set of capacity zero lying on the boundary of D. If D is mapped one-to-one and conformally onto the unit-circle, then the set of points which corresponds to E is of capacity zero.

3. R. Nevanlinna[9] has proved: if a one-valued meromorphic function defined in the whole finite plane except for a bounded closed set of capacity zero does not take any value of a set of

9) R. Nevanlinna, loc. cit. pp. 134-135.

capacity positive, then the function reduces to a constant. S. Kametani[10] has extended this theorem and proved that a one-valued analytic function defined in a domain except for a bounded closed set of capacity zero of essential singularities takes all values except those belonging to a set of capacity zero in the neighbourhood of its essential singularity. In other words, if a one-valued analytic function defined in a domain except for a closed set of capacity zero lying inside of this domain does not take any value of a set of capacity positive, then this function is meromorphic in the whole domain.

Nevanlinna's extension of the principle of maximum is as follows: If $\varphi(z)$ is regular and bounded in a domain D and there exists $\overline{\lim}\,|\varphi(\zeta)| \leqq M$ for every boundary point ζ except for those belonging to a set of capacity zero, then $|\varphi(z)| < M$ in D.

We have as extensions of these theorems to a class of pseudo-regular functions the following theorems.

Theorem 3. Let D be a finite domain on the z-plane bounded by a Jordan curve C and a closed set E of capacity zero lying inside C and $w=f(z)$ be one-valued and pseudo-regular in D and on C and have an essential singularity at every point of E. Then the function $f(z)$ takes all values except perhaps those contained in a set of capacity zero, if the integral

$$\int^{\infty}\frac{du}{C(u)} \tag{1}$$

diverges, where $C(u)$ is the maximum of the 'Dilatationsquotient' of $f(z)$ on the niveau curve C_u of the potential of Evans' theorem.

Proof. Let W be the Riemann surface onto which the domain D is mapped by $w=f(z)$. By the theorem of conformal representation, the interior of W can be transformed one-to-one and conformally on a finite domain G on the ζ-plane by a function $\zeta = g(w)$. Let $w = h(\zeta)$ be the inverse function of $g(w)$.

Since the transformation of W on G is conformal, the function $\zeta = \varphi(z) = g(f(z))$ is pseudo-regular in D and $D_{z|w} = D_{z|\zeta}$ and the domain D is transformed pseudo-conformally on the domain G by $\zeta = \varphi(z)$. Let L be the image of C, then G is a domain bounded by L and a closed set F lying inside L. Since the integral (1) diverges, F is a set of capacity zero by Theorem 1.

Let z_0 be an arbitrary point of E. Since E is a capacity zero, there exists a descending sequence $\{r_n\}$ of radii tending to zero such that the circle C_n; $|z-z_0| = r_n$ does not interesect with E. Let $\varDelta_n$ be the domain bounded by the image of c_n on the ζ-plane.

10) S. Kametani, Proc. Imp. Acad. Jap. 17 (1941), M. Tsuji, loc. cit. T. Yosida, Sijo Sugaku Danwakai, 2. 9. (1948).

Since $\varDelta_n$ decreases monotonically, the set $\varDelta = \pi\bar{\varDelta}_n$ consists of a single point or is a continum more than one point, where $\bar{\varDelta}_n$ is the closure of $\varDelta_n$. As $\varDelta_n$ tends to a part of F and F is of capacity zero, $\varDelta$ consists of a single point. Hence every point of E is not a point of essential singularity of $\varphi(z)$.

We assume that $f(z)$ does not take any value of a set M of capacity positive. Since $h(\zeta)$ is one-valued and regular and does not take any value of M and F is of capacity zero, $h(\zeta)$ is meromorphic at every points of F by the theorem of Nevanlinna-Kametani, so that the function $f(z) = h(\varphi(z))$ has no essential singularity on E, which contradicts the hypothesis. Hence $f(z)$ takes every values except those belonging to a set of capacity zero.

If E consists of only a point, then F also consists of only a point by the corollary of Theorem 1. Since $h(\zeta)$ is one-valued, analytic and does not take any value omitted by $f(z)$. By the theorem of Picard we have the theorem of Lavrentieff as the corollary of this theorem.

Corollary. A pseudo-regular function takes every value infinitely many times except possibly at most two in any neighbourhood of its essential singularity $z = z_0$, if the integral

$$\int_0^{r_0} \frac{dr}{rD(r)}$$

diverges, where $D(r)$ is the maximum of the 'Dilatationsquotient' of $f(z)$ on the circle $|z-z_0| = r$.

By the well known method we have the following precisement of Theorem 3.

Theorem 4. Under the same condition as Theorem 3, the function $f(z)$ takes every values infinitely many times except those belonging to a set of capacity zero.

Theorem 5. Let the function $f(z)$ be bounded and pseudo-regular in a simply connected domain D. If there exists

$$\varlimsup_{z \to z_0} |f(z)| \leqq M$$

at every point z_0 on the boundary of D except at those belonging to a set E of capacity zero and the integral

$$\int^{\infty} \frac{du}{C'(u)} \tag{2}$$

diverges, then we have

$$|f(z)| \leqq M$$

in D, where $C'(u)$ is the maximum of the 'Dilatationsquotient' of $f(z)$ on the common part C'_u of D and the niveau curve C_u of the potential of Evans' theorem.

Proof. Let W be the Riemann surface onto which the domain D is mapped one-to-one ane pseudo-conformally by $w=f(z)$. W can be mapped one-to-one and conformally onto the unit-circle $|\zeta|<1$ by $\zeta=g(w)$. Put

$$\zeta = h(z) = g(f(z))$$

and consider the transformation by $\zeta = h(z)$.

Since $D_{z|w} = D_{z|\zeta}$ and the integral (2) diverges, the set F of points on $|\zeta|=1$ which corresponds to E is of capacity zero by Theorem 2. Let $w=\varphi(\zeta)$ be the inverse function of $g(w)$. Since $\varphi(\zeta)=f(z)$ and $\varphi(\zeta)$ is regular in $|\zeta|<1$, we have $|\varphi(\zeta)|\leqq M$ in $|\zeta|<1$ by Nevanlinna's extension of the principle of maximum. Hence we have $|f(z)|<M$ in D.