118. *Complete Continuities of Linear Operators.*

By Masahiro NAKAMURA.
Osaka Normal College, Tennoji, Osaka.
(Comm. by K. KUNUGI. M.J.A., Nov. 12, 1951.)

1. A linear operator T mapping a Banach space E into itself is, according to F. Riesz (K. Yosida [14), *(weakly) completely continuous* provided that T carries the unit sphere into a (weakly) compact set. These operators, as it is well-known, play an important role in the theory of abstract integral equations (mean ergodic theorems).

Concerning weakly completely continuous operators, V. Gantmacher [5] proved, corresponding to Schauder's Theorem, that both T and its conjugate operator T^* are weakly completely continuous. He shows further, assuming a separability condition, the weak complete continuity is derivable by a condition concerning the values of its second conjugate T^{**}.

The first aim of the present note is to show that Gantmacher's conditions are equivalent without assuming the separability. This is carried out in Theorem 1 with the use of the Moore-Smith convergence of elements of Banach spaces which is introduced by L. Alaoglu [1]: A phalanx or a directed set of $x_\alpha \varepsilon E$ ($f_\alpha \varepsilon E^*$) *converges weakly (weakly*)* to $x \varepsilon E$ ($f \varepsilon E^*$) if and only if $\{x_\alpha\}$ ($\{f_\alpha\}$) is bounded and $f(x_\alpha)$ ($f_\alpha(x)$) converges in the sense of Moore-Smith to $f(x)$ for all $f \varepsilon E^*$ ($x \varepsilon E$), where E^* will mean the conjugate space of E. This convergence will determine a topology of E (E^*) (cf. Alaoglu [1], Bourbaki [3], Tukey [13]). It will be called this topology as the *weak (weak*) topology* of E (E^*). It is known that *the unit sphere of the conjugate space is compact with respect to the weak* topology* (cf. Alaoglu [1], Bourbaki [2], Kakutani [7]).

In the connection with Gantmacher's Theorem, a similar formulation for strong complete continuity will be expected. It is possible to do combining the theorems due to J. Schauder [12] and I. Gelfand [6], and will be formulated in Theorem 2. Although the proof is already known, it will be given a short proof, basing on a compactness theorem due to I. Gelfand [6]. (The proof here employed, including that of the compactness theorems of Gelfand and Phillips, is taken from a letter of Shûichi Takahashi, who send it to the author in the middle of 1949). It is to be noted that his proof of Schauder's theorem is closedly connected in some sense to a recently published proof of S. Kakutani [8]. The author expresses his thanks to S. Takahashi for the permission of the publication in the present note).

2. Gantmacher's theorem will be slightly sharpened in the following form:

THEOREM 1. *Concerning a linear operator T, the following conditions are equivalent:*

(1) *T is weakly completely continuous,*

(2) *T^* is weakly completely continuous,*

(3) *T^* maps a weakly* converging phalanx into weakly converging one,*

(4) *T^{**} maps the second conjugate space E^{**} into E.*

Proof. It will be shown that (1) imples (4) implies (3) implies (2) implies (1).

(1) implies (4); Let $\ddot{x}$ be an arbitrary element of E^{**}. By Helly's Theorem (cf. Kakutani [7] in which an elegant proof of Mimura is contained), there exists a phalanx $\{x_\alpha\}$ which converges weakly* to $\ddot{x}$ in E^{**}. Since T^{**} is continuous with respect to the weak* topology of E^{**}, $x_\alpha T = x_\alpha T^{**}$ converges weakly* to $\ddot{x}T^{**}$. Since $\{x_\alpha\}$ is bounded and T is weakly completely continuous, $\{x_\alpha T\}$ lies in a weakly compact set of E. In E, the weak topology coincides with the relative topology of the weak* topology as a subset of E^{**}, whence if $\{x_\alpha T\}$ has a cluster point x in the weak topology of E then x is also a cluster point in the weak* topology of E^{**}. This shows that $\ddot{x}T^{**}$ belongs to E.

(4) implies (3): If f_α converges weakly* to 0, $\ddot{x}(f_\alpha T^*) = \ddot{x}T^{**}(f_\alpha)$ converges to 0, since $\ddot{x}T^{**}$ belongs to E. Hence $f_\alpha T^*$ converges weakly to 0. This shows T^* is continuous with respect to the weak* topology and the weak topology of E^*.

(3) implies (2): Since the unit sphere of the conjugate space is compact in the weak* topology, this is obvious by the hypothesis.

(2) implies (1): By the above arguments, the weak complete continuity of T^* implies that of T^{**}. Hence the unit sphere of the second conjugate space is mapped into a weakly compact set C of E^{**}. By this mapping, the unit sphere of E mapped into a closed subset D of C. Since E is closed with respect to the weak topology of E^{**} (This is a consequence of the fundamental theorem of S. Mazur concerning the convex sets of Banach spaces), D is included in E, whence D is compact in the weak topology of E^{**}, and so weakly compact in E. This completes the proof of the theorem.

It will be noted that the simplification of Gantmacher's proof is (although it was not used explicitly) supported by a theorem of W. F. Eberlein [4] which proved the equivalence of weak precompactness and weak sequential compactness in Banach spaces.

The characterization of weakly completely continuous operators in means of (4) has some applications. Here it will be listed a few of them. Firstly, a theorem of S. Mazur [10], concerning the numbers of linearly independent solutions of functional equations $x-xT=0$ and $f-fT^*=0$, follows from (4) for weakly completely continuous T. It is to be noted, that Mazur's theorem follows from the mean ergodic theorem for a weakly completely continuous T with $|T|=1$. Nextly, an application of (4) is possible for the representation problems of linear operators acting on concrete Banach spaces. For an example, a linear operacor T on (c) is weakly completely continuous if T^{**} carries (m) into (c). The precise condition can be obtained using the representation of linear functionals on (m) (cf. S. Kakutani and the author [9]), and so the details will be omitted.

3. An analogue of Theorem 1 for strongly completly continuous operators is as follows:

THEOREM 2 (GELFAND-SCHAUDER). *For a linear operator T, the following conditions are equivalent:*

(1) *T is completely continuous,*

(2) *T^* is completely continuous,*

(3) *T^* is continuous as a function on the unit sphere of the conjugate space with the weak* topology and the range with the strong topology.*

Naturally, (3) can be given in the following form:

(3′) *If f_α converges weakly* to 0, $f_\alpha T^*$ converges strongly to 0.*

We shall give a proof basing on the idea of S. Takahashi. It requires some lemmas.

LEMMA 1. *Let M and N be a precompact space in the sense of N. Bourbaki [3], $C(M)$ and $C(N)$ be the space of all uniformly continuous functions on M and N respectively with the usual norm. Furthermore, suppose that T is a mapping on M with range in $C(N)$, and that "conjugate" T^* is defined by $Tx(f)=T^*f(x)$ where $x\varepsilon M$ and $f\varepsilon N$. Then the following conditions are equivalent:*

(1) *T is uniformly continuous,*

(2) *T^* is uniformly continuous,*

(3) *TM is precompact in $C(N)$,*

(4) *T^*N is precompact in $C(M)$.*

The proof of this lemma is easy, and so omitted. It is a simple consequence of Arzela's Theorem.

LEMMA 2. *A uniformly bounded set M in a Banach space is precompact if and only if $f(x)$ defines a continuous mapping from the unit sphere of the conjugate space with the weak* topology and into $B(M)$, the Banach space of all uniformly bounded functions on M.*

Setting N as the unit sphere of the conjugate space, Lemma 2 becomes a direct consequence of the above Lemma 1. Indicated mapping becomes T^*, since $f(x)$ defines a mapping from M to $C(N)$. The topology of M will be given by the weak topology.

THEOREM 3 (GELFAND-PHILLIPS). *Concerning a bounded set M of a Banach space, the following conditions are equivalent:*

(1) *M is precompact,*

(2) *$f_\alpha(x)$ converges uniformly on M whenever f_α converges weakly* to* 0.

(3) *$f(x)$ defines a completely continuous transformation from E^* to $B(M)$.*

The equivalences (1) and (2), (1) and (3) are versions of Lemma 2.

PROOF OF THEOREM 2. It will be proved that (1) implies (3) implies (2) implies (1).

(1) implies (3): Let S be a unit sphere of the Banach space, and let f_α be a phalanx converging weakly* to 0. Then $f_\alpha T^*(x) = f_\alpha(xT)$ converging uniformly for all x belonging to S since ST is precompact and Theorem 3 is applicable. This shows the metrical convergence of $f_\alpha T^*$.

(3) implies (2): By the compactness of the unit sphere of the conjugate space with respect to the weak* topology, the image by T^* is obviously compact.

(2) implies (1): By the above arguments, the hypothesis implies the complete continuity of T^{**}. This yields the strong closedness of E in E^{**} the strong complete continuity of T.

References.

1) L. Alaoglu: *Weak topologies of normed linear spaces*, Ann. of Math., 41 (1940), 252-267.

2) N. Bourbaki: *Sur les espace de Banach*, C. R., 206 (1938), 1701-1704.

3) N. Bourbaki: Topologie generale, Paris, 1940.

4) W. F. Eberlein: *Weak compactness in Banach spaces*, Proc. Nat. Acad. Sci., 33 (1947), 51-53.

5) V. Gantmacher: *Ueber schwache totalstetige Operatoren*, Mat. Sbornik, 7 (1940), 301-307.

6) I. Gelfand: *Abstrakte Funktionen und lineare Operationen*, ibid., 4 (1938), 235-284.

7) S. Kakutani: *Weak topology and regularity of Banach spaces*, Proc. Japan Acad., 15 (1931), 169-173.

8) S. Kakutani: *A proof of Schauder's theorem*, J. of Math. Soc. Jap., 3 (1951), 228-231.

9) S. Kakutani and M. Nakamura: *Banach limit and the Cech compactification of a countable discrete set*, Proc. Japan Acad., 19 (1943), 224-229.

10) S. Mazur: *Ueber Nullstellen linearer Operationen*, Stud. Math., 2 (1930), 11-20.

11) R. S. Phillips: *On linear transformations*, Trans. Amer. Math. Soc., 48 (1941), 516-541.

12) J. Schauder: *Ueber lineare vollstetige Funktionaloperationen*, Stud. Math., 2 (1930), 183-196.

13) J. W. Tukey: Convergence and uniformity in topology, Princeton, 1940.

14) K. Yosida: *Mean ergodic theorem in Banach spaces*, Proc. Japan Acad., 14 (1938), 292-294.