112. *On an Example of a Measure Preserving Transformation Which is Not Conjugate to Its Inverse.*

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1. **Introduction.** If G is a symmetric group it is evident that any element T of G is conjugate to its inverse T^{-1}, because T and T^{-1} are decomposed into the same types of cycles. If the group G of all measure preserving transformations on a measure space of Lebesgue type is regarded as a generalization of symmetric groups there arises naturally the question whether any measure preserving transformation is conjugate to its inverse or not. If the measure preserving transformation under consideration is ergodic and has pure point spectrum, it was shown by J. von Neumann that the invariant for its conjugate class was the additive group of the eigenvalues.[1] Therefore for such a transformation the question is answered affirmatively.[2] But generally the question has remained open, because any general invariants for the conjugate classes have not yet been found. We obtained a criterion to be conjugate to each other for some ergodic transformations of special types in our previous paper "Ergodic Skew Product Transformations on the Torus".[3] By using a similar criterion to this and by the same method as the one in §7 of E.S. we shall give an example of a measure preserving transformation T which is not conjugate to its inverse. In this sense this paper is a continuation of the previous paper E.S..

We shall not evade to repeat some of the definitions and the results which appeared already in E.S.

2. **The criterion that T is conjugate to T^{-1}.** Let X and Y be the usual Lebesgue measure spaces of the circles with unit length. We denote by m the Lebesgue measure on X. The direct product measure space Ω of X and Y is the usual Lebesgue measure space of the two-dimensional torus. Let γ be an irrational number on X and let $\alpha(x)$ be a measurable mapping from X into Y. Then the transformation T on Ω which is defined in the following way is called a skew product measure preserving transformation:

$$T(x, y) = (x+\gamma,\ y+\alpha(x)).$$

$\alpha(x)$ is called the α-function of the skew product transformation T.

1) "Zur Operatorenmethode in der klassischen Mechanik" Ann. of Math. **33** (1932).

2) P.R. Halmos and J. von Neumann: Operator methods in classical mechanics II, Ann. of Math. **43** (1942).

3) Osaka Math. J. Vol. 3, No. 1 (1951). We shall refer to this paper as E.S.

Theorem 1. *Let T be an ergodic skew-product transformation. T is conjugate to its inverse if and only if there exists an element v of X and a measurable mapping $\theta(x)$ from X into Y such that*

$$\alpha(-x+v)\pm\alpha(x)=\theta(x)-\theta(x+\gamma) \tag{1}$$

holds.

Proof. T^{-1} is given by

$$T^{-1}(x,\ y)=(x-\gamma,\ y-\alpha(x-\gamma)).$$

If the equality (1) holds, put

$$V(x,\ y)=(-x+v+\gamma,\ \pm y+\theta(x)).$$

Then we can easily verify that V is a measure preserving transformation on Ω and that $VTV^{-1}=T^{-1}$ holds.

Conversely if T and T^{-1} are coujugate, then let V be a measure preserving transformation on Ω which gives the equality $VTV^{-1}=T^{-1}$. Put

$$V(x,\ y)=(g(x,\ y),\ h(x,\ y)).$$

Then by the same argument as in the proof of Theorem 3 in §3 of E.S. we have

$$g(x,\ y)=-x+v, \qquad h(x,\ y)=\pm y+\theta(x).$$

From these we obtain easily the equality (1).

3. The construction of the desired example. Let γ be an irrational number defined by the continued fraction:

$$\gamma=\cfrac{1}{k_1+\cfrac{1}{k_2+\ddots+\cfrac{1}{k_n+\ddots}}} \tag{2}$$

Put $\delta_0=1$, $\delta_1=\gamma$, then we have the sequence of the positive quantities $\{\delta_n\}$, $n=1, 2,\dots$ successively by the following division-process:

$$\delta_0=k_1\delta_1+\delta_2,\ \delta_1=k_2\delta_2+\delta_3,\ \dots,\ \delta_{n-1}=k_n\delta_n+\delta_{n+1}.$$

Let $\{p_n\}$ be the sequence of the integers defined by

$$p_n=k_np_{n-1}+p_{n-2}. \tag{3}$$

It is evident that p_n is a function of $k_1, k_2, \dots, k_n$, we use as usual the following notation:

$$p_n=[k_1,\ k_2,\ \dots,\ k_n].$$

It is easily verified that the following equality holds

$$\delta_i=[k_{i+1},\ \dots,\ k_n]\,\delta_n+[k_{i+1},\ \dots,\ k_{n-1}]\,\delta_{n+1} \quad \text{for } 0\leqq i\leqq n-2. \tag{4}$$

Especially for $i=0$ we have

$$1 = p_n \delta_n + p_{n-1}\delta_{n+1}. \tag{5}$$

For $i=n-1$ we have

$$\delta_{n-1} = k_n \delta_n + \delta_{n+1}. \tag{6}$$

Put $m_n = p_n + p_{n-1}$, and let M_n be the set of all points of the form $s\gamma$ on the circle X, where s is any integer such that $0 \leqq s \leqq m_n - 1$.

Let N_n be the set of intervals whose endpoints belong to M_n and whose inner points never belong to M_n. Then in Lemma 2 in §7 of E.S. it is shown that all intervals of N_n are divided into the two classes $\varDelta_n$ and $\varGamma_n$, the former class being the set of intervals of the length δ_n and the latter class being the set of intervals of the length δ_{n+1}. Further it is shown in the same lemma that the number of successive intervals of $\varDelta_n$ is k_n or k_n+1 and intervals of $\varGamma_n$ are isolated. As a whole $\varDelta_n$ contains p_n intervals and $\varGamma_n$ contains p_{n-1} intervals.

For any interval E on X let us denote by $\nu_n(E)$ the number of the points of M_n which fall on E.

Let the sequence of positive integers $\{k_n\}$ which defines the irrational number γ by (2) satisfy the following conditions:

(7) Every k_n is divided by 8,

$$\lim_{n\to\infty} \frac{[k_1, k_2, \ldots, k_{n-1}]}{k_{n+1}} = 0, \tag{8}$$

$$\lim_{n\to\infty} \frac{k_{n-1} + \sum_{i=1}^{n-2} k_i [k_{i+1}, \ldots, k_{n-1}]}{k_{n+1}} = 0. \tag{9}$$

Let l be the number defined by

$$l = \sum_{i=1}^{\infty} \frac{k_i \delta_i}{8}. \tag{10}$$

It is easy to verify that l is smaller than 1/4.

Lemma. $l = q_n \delta_n + \frac{\delta_n}{8} + \varepsilon_n$, where q_n is an integer and $\lim_{n\to\infty} \frac{\varepsilon_n}{\delta_n} = 0$ holds.

Proof. $l = \sum_{i=1}^{n+1} \frac{k_i \delta_i}{8} + \sum_{i=n+2}^{\infty} \frac{k_i \delta_i}{8}$.

$$\sum_{i=n+2}^{\infty} \frac{k_i \delta_i}{8} < \frac{\delta_{n+1}}{8} + \frac{\delta_{n+2}}{8} + \ldots < \frac{\delta_{n+1}}{8} + \frac{1}{2}\frac{\delta_{n+1}}{8} + \frac{1}{2^2}\frac{\delta_{n+1}}{8} + \ldots = \frac{\delta_{n+1}}{4}.$$

By (4) and (6) we have

$$\sum_{i=1}^{n+1} \frac{k_i \delta_i}{8} = \frac{1}{8}\sum_{i=1}^{n-2} k_i \delta_i + \frac{k_{n-1}\delta_{n-1}}{8} + \frac{k_n \delta_n}{8} + \frac{k_{n+1}\delta_{n+1}}{8}$$

$$= \frac{1}{8}\sum_{i=1}^{n-2} \{k_i [k_{i+1}, \ldots, k_n]\,\delta_n + k_i [k_{i+1}, \ldots, k_{n-1}]\,\delta_{n+1}\}$$

$$+\frac{1}{8}k_{n-1}(k_n\delta_n+\delta_{n+1})+\frac{k_n\delta_n}{8}+\frac{k_{n+1}\delta_{n+1}}{8}$$
$$=\frac{1}{8}\{k_n(k_{n-1}+1)+\textstyle\sum_{i=1}^{n-2}k_i[k_{i+1},\dots,k_n]\}\delta_n$$
$$+\frac{1}{8}\{k_{n-1}+\textstyle\sum_{i=1}^{n-2}k_i[k_{i+1},\dots,k_{n-1}]\}\delta_{n+1}+\frac{k_{n+1}\delta_{n+1}}{8}.$$

By (7) $$q_n=\frac{1}{8}\{k_n(k_{n-1}+1)+\textstyle\sum_{i=1}^{n-2}k_i[k_{i+1},\dots,k_n]\}$$

is an integer and we have by (9)

$$\lim_{n\to\infty}\frac{1}{8}\{k_{n-1}+\textstyle\sum_{i=1}^{n-2}k_i[k_{i+1},\dots,k_{n-1}]\}\frac{\delta_{n+1}}{\delta_n}$$
$$=\lim_{n\to\infty}\frac{1}{8}\frac{k_{n-1}+\sum_{i=1}^{n-2}k_i[k_{i+1},\dots,k_{n-1}]}{k_{n+1}}=0.$$

Further we have

$$\lim_{n\to\infty}\frac{1}{8}\frac{k_{n+1}\delta_{n+1}}{\delta_n}=\lim_{n\to\infty}\frac{1}{8}\frac{k_{n+1}\delta_{n+1}}{k_{n+1}\delta_{n+1}+\delta_{n+2}}=\frac{1}{8}.$$

The proof of the lemma is completed.

We define the intervals E and F on X and the function $\alpha(x)$ as follows

$$E=[l,\,2l],\qquad F=[3l,\,4l],$$

$$\alpha(x)=\begin{cases}\rho & \text{if } x\in E,\\ 2\rho & \text{if } x\in F,\\ 0 & \text{if } x\notin E\cup F,\end{cases}\tag{11}$$

where ρ is any irrational number on Y.

Let T be the skew product transformation with the α-function $\alpha(x)$ defined by (11).

Theorem 2. T is ergodic.

Proof. Suppose that T is not ergodic. Then by Theorem 2 in §2 of E.S. there exists a non-zero integer p and a measurable mapping $\theta(x)$ from X into Y such that

$$p\alpha(x)=\theta(x)-\theta(x+\gamma)\tag{12}$$

holds. From (12) we have for any n

$$p\textstyle\sum_{j=0}^{m_n-1}\alpha(x+j\gamma)=\theta(x)-\theta(x+m_n\gamma).\tag{13}$$

If we choose a suitable subsequence of $\{m_n\}$ the right hand side of (13) converges to a certain function almost everywhere. We shall show that for any subsequence of $\{m_n\}$ the left hand side of (13) never converges almost everywhere. By (11) we have

$$\textstyle\sum_{j=0}^{m_n-1}\alpha(x+j\gamma)=\rho\,\nu_n(E-x)+2\rho\,\nu_n(F-x).\tag{14}$$

Let I_n be the sum of the intervals of Γ_n. Then we have

$$m(I_n)=p_{n-1}\delta_{n+1}.$$

By the condition (8) we have

$$(15)\qquad \lim_{n\to\infty}\frac{m(I_n)}{\delta_n}=\lim_{n\to\infty}\frac{p_{n-1}\,\delta_{n+1}}{\delta_n}=\lim_{n\to\infty}\frac{[k_1,k_2,\dots,k_{n-1}]}{k_{n+1}}=0\,.$$

Put $\quad J_n=(l-I_n)\smile(2l-I_n)\smile(3l-I_n)\smile(4l-I_n).$ [4]

Let Λ_n be the set of intervals of Δ_n which have no common points with J_n. Since Δ_n contains p_n intervals and Γ_n contains p_{n-1} intervals Λ_n contains at least p_n-8p_{n-1} intervals, therefore the measure of the sum of the intervals belonging to Λ_n is at least $(p_n-8p_{n-1})\delta_n$, which is verified to tend to 1 as $n\to\infty$ by (3) and by the fact that $\lim_{n\to\infty}k_n=\infty$.

If $[\beta,\beta+\delta_n]$ is an interval of Λ_n, by (14), (15) and the lemma, we can compute the variation of the function $\sum_{j=0}^{m_n-1}\alpha(x+j\gamma)$ on this interval as follows:

$$(16)\qquad \sum_{j=0}^{m_n-1}\alpha(x+j\gamma)=\begin{cases}\sigma & \text{if}\quad \beta<x<\beta+\frac{\delta_n}{8}+\eta_n\\ \sigma+\rho & \text{if}\quad \beta+\frac{\delta_n}{8}+\eta_n<x<\beta+\frac{2\delta_n}{8}+\eta_n'\\ \sigma & \text{if}\quad \beta+\frac{2\delta_n}{8}+\eta_n'<x<\beta+\frac{3\delta_n}{8}+\eta_n''\\ \sigma+2\rho & \text{if}\quad \beta+\frac{3\beta_n}{8}+\eta_n''<x<\beta+\frac{4\delta_n}{8}+\eta_n'''\\ \sigma & \text{if}\quad \beta+\frac{4\delta_n}{8}+\eta_n'''<x<\beta+\delta_n\end{cases}$$

where σ is a constant depending on the interval $[\beta,\beta+\delta_n]$ and by the lemma and (15) η_n, η_n', η_n'', η_n''' are small quantities of higher order than δ_n and by (15) β is a quantity different from a multiple of δ_n by a small amount of higher order than δ_n.

Thus we can see that for no subsequence of $\{m_n\}$ the left hand side of (13) is almost everywhere convergent.

Theorem 3. T is not conjugate to T^{-1}.

Proof. Suppose that T is conjugate to T^{-1}. Since T has been proved to be ergodic, we may apply Theorem 1 to our case. Therefore there exists an element v of X and a measurable mapping $\theta(x)$ from X into Y such that

$$\alpha(-x+v)\pm\alpha(x)=\theta(x)-\theta(x+\gamma)$$

holds. Therefore for any n we have

$$(17)\qquad \sum_{j=0}^{m_n-1}\alpha(-x-j\gamma+v)\pm\sum_{j=0}^{m_n-1}\alpha(x+j\gamma)=\theta(x)-\theta(x+m_n\gamma)$$

4) Errata to E.S. p. 99, line 7 and line 8: read "$(-M_n)\smile(l-M_n)$" in place of "$M_n\smile(M_n-l)$"; p. 99, line 9 and line 10: read "$(-I_n)\smile(l-I_n)$" in place of $I_n\smile(I_n-l)$; p. 99, line 14 and line 16: read "$m\{(-I_n)\smile(l-I_n)\}$" in place of $m\{I_n\smile(I_n-l)\}$.

As in the proof of Theorem 2 it is sufficient for our purpose to show that for no subsequence of $\{m_n\}$ the left hand side of (17) is almost everywhere convergent.

By (11) we have

$$\sum_{j=0}^{m_n-1}\alpha(-x-j\gamma)=\rho\nu_n(-E-x)+2\rho\nu_n(-F-x). \tag{18}$$

As in the proof of Theorem 2 we can show that for sufficiently many intervals $[\beta, \beta+\delta_n]$ of Δ_n the following equality holds, in this case the effect of the reflection $E\to -E$, $F\to -F$ should be noticed.

$$\sum_{j=0}^{m_n-1}\alpha(-x-j\gamma)=\begin{cases}\tau & \text{if } \beta<x<\beta+\frac{4\delta_n}{8}+\lambda_n\\ \tau+2\rho & \text{if } \beta+\frac{4\delta_n}{8}+\lambda_n<x<\beta+\frac{5\delta_n}{8}+\lambda_n'\\ \tau & \text{if } \beta+\frac{5\delta_n}{8}+\lambda_n'<x<\beta+\frac{6\delta_n}{8}+\lambda_n''\\ \tau+\rho & \text{if } \beta+\frac{6\delta_n}{8}+\lambda_n''<x<\beta+\frac{7\delta_n}{8}+\lambda_n'''\\ \tau & \text{if } \beta+\frac{7\delta_n}{8}+\lambda_n'''<x<\beta+\delta_n\end{cases} \tag{19}$$

where τ is a constant depending on the interval $[\beta, \beta+\delta_n]$ and λ_n, λ_n', λ_n'', λ_n''' are small quantities of higher order than δ_n.

$\sum_{j=0}^{m_n-1}\alpha(-x-j\gamma+v)$ is a translation by v of $\sum_{j=0}^{m_n-1}\alpha(-x-j\gamma)$. If we compare (16) and (19) we see that any translation of (19) cannot cancel the bumps in (16), which proves the theorem.

Remark. The concrete ergodic transformations given in E.S. are conjugate to their inverses; for example if $\alpha(x)=mx$, where m is any integer, then $\alpha(-x+v)+\alpha(x)=\theta(x)-\theta(x+\gamma)$ holds for $v=0$, $\theta(x)=0$, if $\alpha(x)=\rho C_E(x)$, where E is an interval $[a, b]$, and ρ is an irrational number then $\alpha(-x+v)-\alpha(x)=\theta(x)-\theta(x+\gamma)$ holds for $v=a+b$, $\theta(x)=0$.