28. *Probability-theoretic Investigations on Inheritance. VII$_4$. Non-Paternity Problems.*

By Yûsaku KOMATU.

Department of Mathematics, Tokyo Institute of Technology and
Department of Legal Medicine, Tokyo University.

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4. Generalization to case of mixed combinations.

We now attempt to generalize the results obtained in the preceding sections. We succeed here to the notations introduced in § 2 of IV; especially, let the distributions of inherited character in polulations to which mothers and fathers belong be denoted by $\{p_i\}$ and $\{p'_i\}$ $(i=1, \ldots, m)$, respectively. Practical case of proving non-paternity against a given mixed mother-child combination will occur almost exclusively in such a manner that a putative father belongs to the same population as that of true father. We shall therefore restrict our discussions to such cases alone.

Let us now denote by

$$(4.1) \qquad V'(ij; hk)$$

the quantity corresponding to (2.1). The quantity corresponding to $\pi(ij; hk)$ in (1.1) of IV being $\pi'(ij; hk)$ as already introduced in (2.2) of IV, there now appears the quantity

$$(4.2) \qquad P'(ij; hk) = \pi'(ij; hk) V'(ij; hk)$$

instead of (2.2). Further, corresponding to (2.3), (2.4), (2.5), we have

$$(4.3) \qquad P'(ij) = \sum_{h,k} P'(ij; hk),$$

$$(4.4) \qquad P'(ij)/\bar{A}_{ij},$$

$$(4.5) \qquad P' = \sum_{i,j} P'(ij) = \sum_{i,j,h,k} P'(ij; hk);$$

respective summation extending over all possible sets of suffices indicated.

Making use of the convention introduced in (1.10), it is evident, by definition, that the interrelation

$$(4.6) \qquad V'(ij; hk) = [V(ij; hk)]'$$

holds good. On the other hand, as already noticed in (2.15) of IV, we have

$$(4.7) \qquad \pi'(ij; hk) = p_i p_j [\pi(ij; hk)/p_i p_j]'.$$

By means of these relations, we get fundamental relations stating that

$$(4.8)\qquad P'(ij;hk)=p_ip_j[P(ij;hk)/p_ip_j]',$$

$$(4.9)\qquad P'(ij)=p_ip_j[P(ij)/p_ip_j]'.$$

For instance, we obtain, corresponding to (2.14) and (2.18),

$$(4.10)\qquad P'(ii)=p_i^2(1-2S'_2+S'_3),$$

$$(4.11)\qquad P'(ij)=p_ip_j(2(1-2S'_2+S'_3)-4p'_ip'_j+3p'_ip'_j(p'_i+p'_j))\quad (i\neq j),$$

whence it follows a formula representing the *whole probability* of proving non-paternity against given mixed mother-child combinations. In fact, we get

$$\begin{aligned}P'&=\sum_{i=1}^{m}P'(ii)+\sum_{i,j}{}'P'(ij)\\&=S_2(1-2S'_2+S'_3)+\sum_{i,j=1}^{m}p_ip_j\left(1-2S'_2+S'_3-2p'_ip'_j+\frac{3}{2}p'_ip'_j(p'_i+p'_j)\right)\\&\qquad-\sum_{i=1}^{m}p_i^2(1-2S'_2+S'_3-2p'^2_i+3p'^3_i),\end{aligned}$$

i.e., by taking the notations introduced in (1.12) into account,

$$(4.12)\qquad P'=1-2S'_2+S'_3-2S_{1,1}+2S_{2,2}+3S_{1,1}S_{1,2}-3S_{2,3}.$$

If, in particular, we put $\{p'_i\}=\{p_i\}$, then the results will reduce to the corresponding ones already discussed in the preceding sections; the relation (1.14) being to be taken into account.

The results on *sub-probabilities* corresponding to (3.1), (3.2), (3.3), (3.5), (3.7) can also be easily derived. We obtain in turn

$$(4.13)\qquad \sum_{i=1}^{m}P'(ii;ii)=\sum_{i=1}^{m}p_i^2p'_i(1-p'_i)^2=S_{2,1}-2S_{2,2}+S_{2,3},$$

$$(4.14)\qquad \begin{aligned}&\sum_{i=1}^{m}\sum_{j\neq i}P'(ii;ij)=\sum_{i=1}^{m}p_i^2(1-2S'_2+S'_3-p'_i(1-p'_i)^2)\\&=S_2-S_{2,1}-2S_2S'_2+2S_{2,2}+S_2S'_3-S_{2,3},\end{aligned}$$

$$(4.15)\qquad \begin{aligned}&\sum_{i,j}{}'(P'(ij;ii)+P'(ij;jj))=\sum_{i,j=1}^{m}p_ip_jp'_i(1-p'_i)^2-\sum_{i=1}^{m}p_i^2p'_i(1-p'_i)^2\\&=S_{1,1}-(2S_{1,2}+S_{2,1})+(S_{1,3}+2S_{2,2})-S_{2,3},\end{aligned}$$

$$(4.16)\qquad \begin{aligned}&\sum_{i,j}{}'P'(ij;ij)=\frac{1}{2}\sum_{i,j=1}^{m}p_ip_j(p'_i+p'_j)(1-p'_i-p'_j)^2\sum_{i=1}^{m}p_i^2p'_i(1-2p'_i)^2\\&=S_{1,1}-(2S_{1,2}+S_{2,1})-2S_{1,1}^2+(S_{1,3}+4S_{2,2})+3S_{1,1}S_{1,2}-4S_{2,3},\end{aligned}$$

$$(4.17)\qquad \begin{aligned}&\sum_{i,j}{}'\sum_{h\neq i,j}(P'(ij;ih)+P'(ij;jh))\\&=2\sum_{i,j}{}'p_ip_j(1-2S'_2+S'_3-p'_i(1-p'_i)^2-p'_j(1-p'_j)^2)\\&=1-(2S'_2+S_2+2S_{1,1})+(S'_3+4S_{1,2}+2S_{2,1})+2S_2S'_2\\&\qquad-2(S_{1,3}+2S_{2,2})-S_2S'_3+2S_{2,3}.\end{aligned}$$

Addition of (4.13) to (4.17) leads, of course, again to (4.12). Similar results mentioned at the end of the preceding section apply here also. Further, similar table listed there will also be easily constructed, the actual formulation of which may be left to the reader.

5. Illustrative examples.

In order to illustrate the general results obtained in the preceding sections by examples, we first consider the simplest non-trivial case realized by *MN blood type*. The result on pure mother-child combination is classical; indeed, denoting, as usual, the relative frequencies of genes M and N by s and t respectively, the whole probability of proving non-paternity is given by the formula

$$P_{MN} = st(1-st). \tag{5.1}$$

This is a special case ($m=2$) of (2.10). It can be rather briefly derived by a direct manner[1]. However, we can more generally specialize the formula (4.12) to case $m=2$ or derive the result on mixed mother-child combination rather briefly by a direct calculation, obtaining

$$P'_{MN} = s't'(1-st), \tag{5.2}$$

whence it follows immediately, by specialization $(s', t')=(s, t)$, again the result (5.1).

We now turn our attention to case where recessive genes may be existent. Probability of proving non-paternity with the aid of phenotypes in such a case can also be obtained essentially in quite a similar manner as in the case already discussed. We have only to sum up all the probabilities of mother-child combinations after multiplying by the respective frequencies of deniable types of putative fathers.

As a concrete example, we consider *ABO blood type*. The result on pure combination is classical, which states indeed[2]

$$\begin{aligned} P_{ABO} &= q(p+r)^4 + p(q+r)^4 + pqr^2(2+p+q) \\ &= p(1-p)^4 + q(1-q)^4 + pqr^2(3-r). \end{aligned} \tag{5.3}$$

We indicate further that the corresponding probability on mixed mother-child combination becomes as follows:

$$\begin{aligned} P'_{ABO} &= (p+r)^2 q'(p'+r')^2 + (q+r)^2 p'(q'+r')^2 \\ &\quad + 2rp'q'r' + pqr'^2(p'+q') \\ &= (1-p)^2 p'(1-p')^2 + (1-q)^2 q'(1-q')^2 \\ &\quad + 2rp'q'r' + pqr'^2(1-r'). \end{aligned} \tag{5.4}$$

1) Cf., for instance, A. S. Wiener, M. Lederer, and S. H. Polayes, Studies in isohemagglutination, IV. On the chances of proving non-paternity; with special referrence to blood groups. Journ. of Immun. **19** (1930), 259-282.

2) Cf., for instance, A. S. Wiener, Blood groups and blood transformation. 2nd ed. (Springfield and Baltimore, 1939), p. 172.

The former result (5.3) is nothing but a special case of (5.4), obtained by putting $(p', q', r') = (p, q, r)$.

The corresponding results on *Q blood type* become

$$P_Q = uv^4, \tag{5.5}$$

$$P'_Q = v^2 u' v'^2. \tag{5.6}$$

We further consider *A_1A_2BO blood type*, i.e., the sub-divided *ABO* blood type in which the gene A is divided into A_1 and A_2. The result for pure mother-child combination becomes as follows; the frequencies of genes O, A_1, A_2, B being denoted, as usual, by r, p_1, p_2, q, respectively:

$$P_{A_1A_2BO} = P_{ABO} + p_1 p_2 R, \tag{5.7}$$

where the residual term is given by

$$\begin{aligned} R = & r^2(p_2 + 4q + 2r) + q((p+r)^2 + 2p_1(p_2 + 2r)) \\ & + (p_2 + 2r)(p_2 + q + r)^2 + q(p_2^2 + 4p_2 r + r^2 - p_1^2 - 2p_1p_2 - 2p_2 r) \\ & + q(p_2 + 4q + 2r)(q + 2r) + q((p_1 + q)(p_2 + q + 2r) + q(p_1 + 2r)) \\ & + q((p_2 + q + r)(2p_2 + 2q + r) + p_2(2q + r) + 3qr) \\ = & 4r^3 + 3(2p_2 + 5q)r^2 + 2(3p_1 q + 2p_2^2 + 9p_2 q + 11q^2)r \\ & \quad + p_2^3 + q(3p_1p_2 + 2p_1 q + 6p_2^2 + 9p_2 q + 7q^2). \end{aligned} \tag{5.8}$$

In view of the relation $p_1 + p_2 + q + r = 1$, it can also be expressed in the form

$$\begin{aligned} R = & (2 - 2p_1 - p_2)(2(q + r)^2 + p_2(p_2 + 2r)) \\ & + q(4p_2 + 6q + 6r - 2p_1p_2 - p_1 q + qr - 3r^2). \end{aligned} \tag{5.9}$$

In conclusion, we supplement the results on *$Qq_\pm$ blood type*, stating

$$P_{Qq\pm} = uv^4 + v_1 v_2^4, \tag{5.10}$$

$$P'_{Qq\pm} = v^2 u' v'^2 + v_2^2 v_1' v_2'^2. \tag{5.11}$$

—To be continued—