52. Probability theoretic Investigations on Inheritance. IX_2. Non-Paternity Concerning Mother-Children Combinations.

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4. Non-paternity against both children separately.

We have discussed hitherto in the present chapter the problem of proving non-paternity, indifferent to a type of first child, against second child at any rate; it has been a matter of indifference whether the proof of non-paternity against first child is possible or not. We now proceed to the problem of proving non-paternity against both children of the same family separately.

For that purpose, we introduce as basic quantities, besides the probability of mother-children combination defined in (3.1) of IV, that of proving non-paternity of a man chosen at random against both children of a fixed triple; namely, given a triple consisting of a mother A_{ij}, her first child A_{hk} and her second child A_{fg}, we ask at how many rate the non-paternity can be established against both first and second children *separately*, i.e., indifferent to types of second and first children respectively. The probability in question be denoted by

(4.1) $$V(ij;\ hk,\ fg).$$

Of course, only the cases are significant where there exist common suffices between i, j and h, k and between i, j and f, g. Thus, *the probability of proving non-paternity against both children separately*, the combination-probability being also taken into account, is then given by

(4.2) $$Q(ij;hk,fg)=\pi(ij;hk,fg)\ V(ij;\ hk,fg).$$

The quantities (4.1) are evidently symmetric with respect to types of both children; namely, we have

(4.3) $$V(ij;hk,fg)= V(ij;fg,\ hk).$$

On the other hand, since the probabilities of mother-children combination possess an analogous symmetry character, as noticed in (3.4) of IV, we see that the quantities in (4.2) also satisfy a symmetry relation of the same nature, i.e.,

(4.4) $$Q(ij;\ hk,fg)=Q(ij;fg,\ hk).$$

Now, if the proof of non-paternity is possible against both children separately, then it is necessarily also possible against a distinguished child among them, and hence the inequality

$$(4.5) \qquad V(ij;\, hk, fg) \leqq V(ij;\, fg)$$

is valid in general. It implies a corresponding inequality between (2.3) and (4.2), namely

$$(4.6) \qquad Q(ij;\, hk, fg) \leqq P(ij;\, hk, fg).$$

It is evident that if, in particular, the types of both children coincide each other, the equality sign holds in (4.5) and hence also in (4.6):

$$(4.7) \qquad V(ij; fg, fg) = V(ij; fg),$$

$$(4.8) \qquad Q(ij; fg, fg) = P(ij; fg, fg).$$

The values of quantities in (4.1) can be found in a similar manner as in case of $V(ij; fg)$ in (2.1) of VIII. The next table concerns the set of deniable types of a man and its probability against each given triple.

Mother	1st child \ 2nd child	A_{ii}	A_{ih}	A_{ik}
A_{ii}	A_{ii}	$A_{ab}(a, b \neq i)$ $(1-p_i)^2$	$A_{ab}(a, b \neq i, h)$ $(1-p_i-p_h)^2$	$A_{ab}(a, b \neq i, k)$ $(1-p_i-p_k)^2$
	A_{ih}	$A_{ab}(a, b \neq i, h)$ $(1-p_i-p_h)^2$	$A_{ab}(a, b \neq h)$ $(1-p_h)^2$	$A_{ab}(a, b \neq h, k)$ $(1-p_h-p_k)^2$
	A_{ik}	$A_{ab}(a, b \neq i, k)$ $(1-p_i-p_k)^2$	$A_{ab}(a, b \neq h, k)$ $(1-p_h-p_k)^2$	$A_{ab}(a, b \neq k)$ $(1-p_k)^2$

Mother	1st child \ 2nd child	A_{ii}	A_{jj}	A_{ij}	A_{ih} A_{jh}	A_{ik} A_{jk}
A_{ij}	A_{ii}	$A_{ab}(a, b \neq i)$ $(1-p_i)^2$	$A_{ab}(a,b \neq i,j)$ $(1-p_i-p_j)^2$	$A_{ab}(a,b \neq i,j)$ $(1-p_i-p_j)^2$	$A_{ab}(a,b \neq i,h)$ $(1-p_i-p_h)^2$	$A_{ab}(a,b \neq i,k)$ $(1-p_i-p_k)^2$
	A_{jj}	$A_{ab}(a,b \neq i,j)$ $(1-p_i-p_j)^2$	$A_{ab}(a,b \neq j)$ $(1-p_j)^2$	$A_{ab}(a,b \neq i,j)$ $(1-p_i-p_j)^2$	$A_{ab}(a,b \neq j,h)$ $(1-p_j-p_h)^2$	$A_{ab}(a,b \neq j,k)$ $(1-p_j-p_k)^2$
	A_{ij}	$A_{ab}(a,b \neq i,j)$ $(1-p_i-p_j)^2$	$A_{ab}(a,b \neq i,j)$ $(1-p_i-p_j)^2$	$A_{ab}(a,b \neq i,j)$ $(1-p_i-p_j)^2$	$A_{ab}(a,b \neq i,j,h)$ $(1-p_i-p_j-p_h)^2$	$A_{ab}(a,b \neq i,j,k)$ $(1-p_i-p_j-p_k)^2$
	A_{ih} or A_{jh}	$A_{ab}(a,b \neq i,h)$ $(1-p_i-p_h)^2$	$A_{ab}(a,b \neq j,h)$ $(1-p_j-p_h)^2$	$A_{ab}(a,b \neq i,j,h)$ $(1-p_i-p_j-p_h)^2$	$A_{ab}(a,b \neq h)$ $(1-p_h)^2$	$A_{ab}(a,b \neq h,k)$ $(1-p_h-p_k)^2$
	A_{ik} or A_{jk}	$A_{ab}(a,b \neq i,k)$ $(1-p_i-p_k)^2$	$A_{ab}(a,b \neq j,k)$ $(1-p_j-p_k)^2$	$A_{ab}(a,b \neq i,j,k)$ $(1-p_i-p_j-p_k)^2$	$A_{ab}(a,b \neq h,k)$ $(1-p_h-p_k)^2$	$A_{ab}(a,b \neq k)$ $(1-p_k)^2$

As seen from the table, the quantities lying on principal diagonals coincide in conformity to (4.7), with the corresponding ones in one child case. With respect to remaining quantities, the inequality sign will appear, in general, except the cases essentially contained in the following equalities:

$$(4.9)\qquad V(ij;ii,ij)=V(ij;ij)=(1-p_i-p_j)^2 \qquad (i\neq j),$$

$$(4.10)\qquad V(ij;ih,jh)=V(ij;jh)=(1-p_h)^2 \qquad (h\neq i,j).$$

The values of V's in (4.1) having thus been determined, those of Q's in (4.2) can immediately be calculated by means of their own definition.

The partial sums with respect to pair of mother and her one child are similarly defined as in (2.4) and (2.5); namely,

$$(4.11)\qquad J(ij;fg)=\sum_{h\leq k}Q(ij;hk,fg),\quad J(ij;hk)=\sum_{f\leq g}Q(ij;hk,fg);$$

because of the symmetry relation (4.4), the function J is the same in both relations (4.11). In view of (4.6), the corresponding inequalities follow, stating

$$(4.12)\qquad J(ij;fg)\leqq P(ij;fg),\qquad J(ij;hk)\leqq I(ij;hk),$$

which are also immediately evident by the meanings of the quantities contained.

We now derive the results concerning the first or rather the second quantity (4.11), which correspond to (2.6) to (2.10). It will be convenient to consider an excess (or rather a deficiency) of (2.5). For a pair of a mother and her first child being of the same homozygote A_{ii}, we get, in view of (4.8), the difference

$$\begin{aligned}J(ii;ii)-I(ii;ii)&=Q(ii;ii,ii)+\sum_{k\neq i}Q(ii;ii,ik)\\&\quad-\Big(P(ii;ii,ii)+\sum_{k\neq i}P(ii;ii,ik)\Big)=-\sum_{k\neq i}(P(ii;ii,ik)-Q(ii;ii,ik))\\&=-\sum_{k\neq i}\tfrac{1}{2}p_i^4p_k(2-p_i-2p_k),\end{aligned}$$

and hence

$$(4.13)\qquad\begin{aligned}J(ii;ii)&=I(ii;ii)-\tfrac{1}{2}p_i^4(2-2S_2-3p_i+3p_i^2)\\&=\tfrac{1}{2}p_i^3(2-2S_2+S_3-2(2-S_2)p_i+4p_i^2-3p_i^3).\end{aligned}$$

We get similarly

$$(4.14)\qquad\begin{aligned}J(ii;ih)&=I(ii;ih)-\Big(\tfrac{1}{2}p_i^3p_h^2(2-2p_i-p_h)+\sum_{k\neq i}\tfrac{1}{2}p_i^2p_h^2p_k(2-p_h-2p_k)\Big)\\&=\tfrac{1}{2}p_i^2p_h(2-2S_2+S_3-2(2-S_2)p_h+4p_h^2-3p_h^3)\qquad(h\neq i),\end{aligned}$$

which remains valid also for $h=i$. In a similar manner, we obtain the following results :

$$(4.15)\qquad\begin{aligned}J(ij;ii)&=I(ij;ii)-\Big(\tfrac{1}{4}p_i^3p_j^2(2-p_i-2p_j)+2\sum_{k\neq i,j}\tfrac{1}{4}p_i^3p_jp_k(2-p_i-2p_k)\Big)\\&=\tfrac{1}{4}p_i^2p_j(2(2-2S_2+S_3)+4(2-S_2)p_i-2p_j\\&\qquad+8p_i^2+p_j^2-6p_i^3+p_ip_j(2p_i+p_j))\qquad(i\neq j),\\J(ij;ij)&=I(ij;ij)-\Big(\tfrac{1}{4}p_i^2p_j^2(1+p_i+p_j)(2-2p_i-p_j)\end{aligned}$$

$$
\begin{aligned}
(4.16)\qquad & +\tfrac{1}{4}p_i^2p_j^2(1+p_i+p_j)(2-p_i-2p_j) \\
& +2\sum_{k\neq i,j}\tfrac{1}{4}p_ip_jp_k(p_i+p_j)^2(2-p_i-p_j-2p_k)\Big) \\
& =\tfrac{1}{2}p_ip_j((2-2S_2+S_3)(p_i+p_j)-2(2-S_2)(p_i^2+p_j^2)-4(2-S_2)p_ip_j \\
& +4(p_i^3+p_j^3)+8p_ip_j(p_i+p_j)-3(p_i^4+p_j^4) \\
& -5p_ip_j(p_i^2+p_j^2)-4p_i^2p_j^2) \qquad (i\neq j),
\end{aligned}
$$

$$
\begin{aligned}
& J(ij;ih)=I(ij;ih)-\Big(\tfrac{1}{4}p_i^2p_jp_h^2(2-2p_i-p_h)+\tfrac{1}{4}p_ip_j^2p_h^2(2-2p_j-p_h) \\
& +\tfrac{1}{4}p_ip_jp_h^2(p_i+p_j)(2-2p_i-2p_j-p_h) \\
(4.17)\qquad & +2\sum_{k\neq i,j,h}\tfrac{1}{4}p_ip_jp_h^2p_k(2-p_h-2p_k)\Big) \\
& =\tfrac{1}{4}p_ip_jp_h(2(2-2S_2+S_3)-4p_ip_j+3p_ip_j(p_i+p_j) \\
& -4(2-S_2-p_ip_j)p_h+8p_h^2-6p_h^3) \qquad (i\neq j;\ h\neq i,j).
\end{aligned}
$$

It would be noticed that $J(ij;ih)$ is symmetric with respect to suffices i and j, i.e., $J(ij;ih)=J(ij;jh)$ provided $h\neq i,j$.

The partial sums corresponding to (2.11) to (2.15) can also be derived as follows:

$$(4.18)\qquad \sum_{i=1}^{m}J(ii;ii)=S_3-2S_4-S_2S_3+2S_5+\tfrac{1}{2}S_3^2+S_2S_4-\tfrac{3}{2}S_6,$$

$$
\begin{aligned}
(4.19)\qquad & \sum_{i=1}^{m}\sum_{h\neq i,j}J(ii,ih)=S_2-S_3-3S_2^2+2S_4+\tfrac{7}{2}S_2S_3-2S_5 \\
& +S_2^3-\tfrac{1}{2}S_3^2-\tfrac{5}{2}S_2S_4+\tfrac{3}{2}S_6;
\end{aligned}
$$

$$
\begin{aligned}
(4.20)\qquad & \sum_{i,j}{}'(J(ij;ii)+J(ij;jj))=S_2-3S_3-\tfrac{3}{2}S_2^2+\tfrac{9}{2}S_4 \\
& +\tfrac{11}{4}S_2S_3-\tfrac{15}{4}S_5-\tfrac{1}{4}S_3^2-\tfrac{1}{2}S_2S_4+\tfrac{3}{4}S_6,
\end{aligned}
$$

$$
\begin{aligned}
(4.21)\qquad & \sum_{i,j}{}'J(ij;ij)=S_2-3S_3-3S_2^2+6S_4+\tfrac{13}{2}S_2S_3-\tfrac{15}{2}S_5 \\
& +S_2^3-\tfrac{3}{2}S_3^2-\tfrac{9}{2}S_2S_4+5S_6,
\end{aligned}
$$

$$
\begin{aligned}
(4.22)\qquad & \sum_{i,j}{}'\sum_{h\neq i,j}(J(ij;ih)+J(ij;jh))=1-6S_2+\tfrac{17}{2}S_3 \\
& +5S_2^2-\tfrac{17}{2}S_4-4S_2S_3+\tfrac{7}{2}S_5-\tfrac{1}{2}S_3^2-S_2S_4+2S_6.
\end{aligned}
$$

The sum of (4.18) and (4.19) gives the sub-probability of proving non-paternity against both children of a homozygotic mother separately:

$$(4.23)\qquad S_2(1-3S_2+\tfrac{5}{2}S_3+S_2^2-\tfrac{3}{2}S_4),$$

while the sum of (4.20), (4.21) and (4.22) gives that of a heterozygotic mother:

$$
\begin{aligned}
(4.24)\qquad & 1-4S_2+\tfrac{5}{2}S_3+\tfrac{1}{2}S_2^2+2S_4+\tfrac{21}{4}S_2S_3-\tfrac{31}{4}S_5 \\
& +S_2^3-\tfrac{7}{4}S_3^2-6S_2S_4+\tfrac{31}{4}S_6.
\end{aligned}
$$

The sum of the last two sub-probabilities (4.23) and (4.24), that is, the total sum of (4.18) to (4.22), yields the *whole probability* of proving non-paternity against both children separately, stating that

$$(4.25)\qquad \begin{aligned} J=1-3S_2+\tfrac{5}{2}S_3-\tfrac{5}{2}S_2^2+2S_4+\tfrac{31}{4}S_2S_3-\tfrac{31}{4}S_5 \\ +2S_2^3-\tfrac{9}{4}S_3^2-\tfrac{15}{2}S_2S_4+\tfrac{31}{4}S_6. \end{aligned}$$

The procedure of determining the probabilities of proving non-paternity against both children separately is to be modified suitably, if there exist dominance relations. As illustrative examples, we give here the whole probabilities in case of various human blood types:

$$(4.26)\qquad J_{ABO}=\tfrac{1}{2}p(1+p)(1-p)^4+\tfrac{1}{2}q(1+q)(1-q)^4+\tfrac{1}{4}pqr^2(4+r+7r^2),$$

$$(4.27)\qquad J_Q=\tfrac{1}{2}u(1+u)v^4,$$

$$(4.28)\qquad J_{Qq\pm}=\tfrac{1}{2}(1+u)(uv^4+v_1v_2^4)+\tfrac{1}{2}(u+v_1)v_1v_2^4.$$

On the other hand, we get, from (4.13) and (4.14),

$$(4.29)\qquad J(ii;ii)+\sum_{h\neq i}J(ii;ih)=p_i^2(1-3S_2+\tfrac{5}{2}S_3+S_2^2-\tfrac{3}{2}S_4).$$

This represents the partial sum with respect to a homozygotic mother A_{ii}. The coefficients of p_i^2 in the right-hand side is nothing but the probability when a mother A_{ii} is fixed. Similarly, we get, from (4.15) to (4.17),

$$(4.30)\qquad \begin{aligned} J(ij;ii)+J(ij;jj)+J(ij;ij)+\sum_{h\neq i,\,j}(J(ij;ih)+J(ij;jh)) \\ =2p_ip_j(1-3S_2+\tfrac{5}{2}S_3+S_2^2-\tfrac{3}{2}S_4-(\tfrac{7}{2}-2S_2)p_ip_j \\ +\tfrac{31}{8}p_ip_j(p_i+p_j)-\tfrac{11}{4}p_ip_j(p_i^2+p_j^2)-\tfrac{9}{4}p_i^2p_j^2) \qquad (i\neq j), \end{aligned}$$

the expression possessing an analogous meaning as (4.29). If we sum up (4.29) or (4.30) with respect to i $(1\leqq i\leqq m)$ or to i and j $(1\leqq i<j\leqq m)$, then we get again (4.23) or (4.24), respectively.

Generalization to mixed case is also possible, while the details will be left to the reader. We state here only an expression of the whole probability generalizing (4.25):

$$(4.31)\qquad \begin{aligned} J'=1-3S_2'+\tfrac{5}{2}S_3'-\tfrac{1}{2}(-2S_2'^2+7S_{1,1}^{\;2})+\tfrac{1}{2}(-3S_4'+7S_{2,2}) \\ +\tfrac{1}{4}(-2S_2'S_{1,2}+31S_{1,1}S_{1,2}+2S_2'S_{2,1})-\tfrac{31}{4}S_{2,3} \\ +2S_2'^2S_2-\tfrac{9}{4}S_{1,2}^{\;2}-\tfrac{1}{2}(11S_{1,1}S_{1,3}+4S_2'S_{2,2})+\tfrac{31}{4}S_{2,4}. \end{aligned}$$

The specialization $\{p_i'\}=\{p_i\}$ leads, of course, to the previous result (4.25) of pure case.

—To be continued—