45. *On the Stability of the Satellite Systems.*

By Yusuke HAGIHARA, M.J.A.

(Comm. April 12, 1952.)

The motion of our satellite, the Moon, was successfully studied by Hill on the basis of his ingenious theory on the periodic solution of the problem[1]. Suppose that the Sun at infinity moves round the Earth as the origin in a circular and uniform motion, and that the Moon moves in the rotating co-ordinate plane with the x-axis pointing always towards the Sun. Hill has chosen a periodic orbit for the Moon in this rotating axes with two arbitrary integration constants as his intermediary orbit for the actual motion of the Moon. The actual orbit has been obtained from this intermediary orbit by the superposition of small periodic terms and possibly of small empirical secular terms which are supposed to be due to the tidal friction in the main part. Thus the principal feature on the stability of the motion of the Moon can be judged by the behavior of the intermediary orbit, provided that the amplitudes of the small periodic terms remain always small within finite limits.

Let x and y be the rectangular axes, the x axis always pointing to the Sun which moves round the Earth as the origin with the mean motion n' in the xy-plane. The equations of motion of the Moon are

$$\frac{dx^2}{dt^2}-2n'\frac{dy}{dt}+\left[\frac{\mu}{r^3}-3n'^2\right]x=0,$$

$$\frac{d^2y}{dt^2}+2n'\frac{dx}{dt}+\frac{\mu}{r^3}y=0,$$

where μ is the sum of the masses of the Moon and the Earth multiplied by the constant of gravitation. The integral, the so-called Jacobi integral, can be found easily:

$$\frac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}=\frac{\mu}{r}+\frac{3}{2}n'^2x^2-C,$$

with the constant of integration C. The left-hand member should be positive or zero. Hence the motion should occur within the region, where

$$\frac{\mu}{(x^2+y^2)^{1/2}}+\frac{3}{2}n'^2x^2-C\geqq 0.$$

1) Hill, Amer. J. Math., **5**, 129 and 245, 1873; Collected Math. Works, Vol. I, 284, 1905; Acta Math., **8**, 1, 1886; Collected Math. Works, Vol. I, 243, 1905.

The behavior of the curve is shown in Fig. 1, according as $(2C)^{3/2} >, =$ or $< 9\mu n'$.

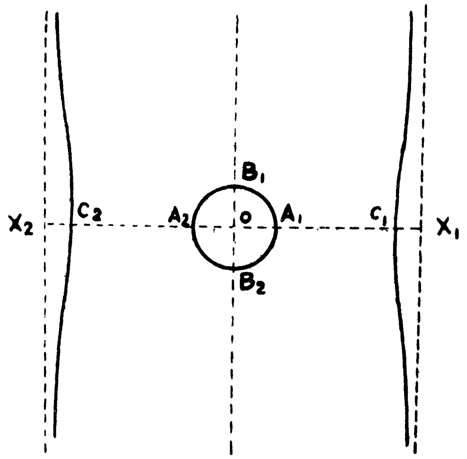

Fig. 1a. $(2C)^{3/2} > 9\mu n'$

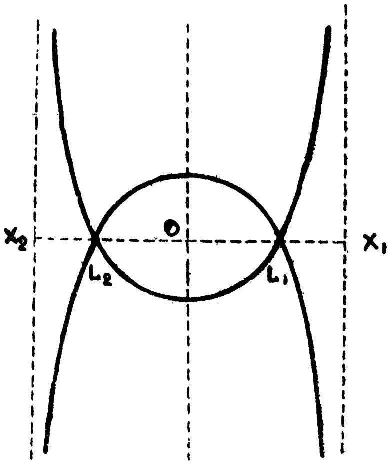

Fig. 1b. $(2C)^{3/2} = 9\mu n'$

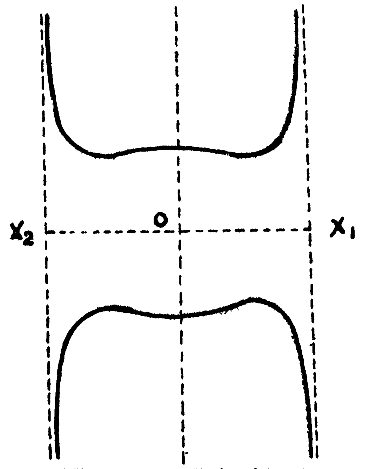

Fig. 1c. $(2C)^{3/2} < 9\mu n'$

In our case $(2C)^{3/2} > 9\mu n'$ and $OX_1 = OX_2 = \sqrt{\dfrac{2C}{3n'^2}} = 500.4992$, $OA_1 = 109.694$, $OA_2 = 109.655$, $OB_1 = OB_2 = 104.408$, the unit being the equatorial radius of the Earth. Our Moon is at present within the oval of zero-velocity $OA_1B_1A_2B_2$. Hence it can not cross the boundary of the oval and always remains inside the oval. Thus the stability of the motion of the Moon is ascertained under the restriction that the deviation of the actual orbit from the intermediary orbit does not grow large enough to escape from the oval.

Similar argument can be applied to other satellite systems with a slight change of the units. Table I gives the change of the units to be made for applying the criterion for the Moon

Table I

Change of Units

Planet	Unit of length	Unit of mass	μ	n'	$9\mu n'$
Earth	$4.27 \cdot 10^{-5}$	$3.036 \cdot 10^{-6}$	$1.160 \cdot 10^{4}$	$1.720 \cdot 10^{-2}$	179.5
Mars	$2.27 \cdot 10^{-5}$	$3.23 \cdot 10^{-7}$	$8.20 \cdot 10^{3}$	$9.16 \cdot 10^{-3}$	675
Jupiter	$4.78 \cdot 10^{-4}$	$9.548 \cdot 10^{-4}$	$2.593 \cdot 10^{3}$	$1.450 \cdot 10^{-3}$	33.83
Saturn	$4.04 \cdot 10^{-4}$	$2.856 \cdot 10^{-4}$	$1.278 \cdot 10^{3}$	$5.81 \cdot 10^{-4}$	6.68
Uranus	$1.662 \cdot 10^{-4}$	$4.373 \cdot 10^{-5}$	$2.82 \cdot 10^{3}$	$2.045 \cdot 10^{-4}$	5.19
Neptune	$1.772 \cdot 10^{-4}$	$5.178 \cdot 10^{-5}$	$2.75 \cdot 10^{3}$	$1.043 \cdot 10^{-4}$	2.585

Table II

Satellite of the Earth

Satellite	n	a	C	$(2C)^{3/2}$	μ/C
Moon	0.230	60.27	111.5	3330	104

Table III

Satellites of Mars

Satellite	n	a	C	$(2C)^{3/2}$	μ/C
I Phobos	19.68	2.76	1486	162000	5.52
II Deimos	4.98	6.92	590	40500	13.9

Table IV

Satellites of Jupiter

Satellite		n	a	C	$(2C)^{3/2}$	μ/C
I	Io	3.55	5.91	218	9110	11.9
II	Europa	1.768	9.40	138	4590	18.77
III	Ganymede	0.878	14.99	86.1	2260	30.1
IV	Callisto	0.376	26.36	49.2	977	52.7
V		12.62	2.53	50.9	1027	51.0
VI		0.02507	160.46	9.06	77.2	286.
VII		0.02421	164.46	8.79	73.8	297.
VIII		0.00850r	329.3	2.70	12.5†	960.
IX		0.00782r	330.4	3.40	17.7†	762.
X		0.0241	162.02	7.45	57.5	348.
XI		0.00908r	316.0	2.89	13.4†	897.
XII		0.01005r	293.2	3.43	17.9†	757.

r : retrograde †: our criterion can not be applied

Table V

Satellites of Saturn

	Satellite	n	a	C	$(2C)^{3/2}$	μ/C
I	Mimas	6.67	3.07	215	8920	5.40
II	Enceladus	4.59	3.94	161	5780	7.20
III	Tethys	3.33	4.88	130	4190	8.93
IV	Dione	2.295	6.24	102	2915	11.36
V	Rhea	1.392	8.72	72.6	1750	15.96
VI	Titan	0.394	20.22	31.3	495	37.05
VII	Hyperion	0.2955	24.49	25.9	373	44.8
VIII	Japetus	0.0793	58.91	11.13	105	121.5
IX	Phoebe	0.0114r	214.4	2.64	12.12	484.
X	Themis	0.301	24.17	26.4	383	43.9

r : retrograde

Table VI

Satellites of Uranus

	Satellite	n	a	C	$(2C)^{3/2}$	μ/C
I	Ariel	2.492	7.71	202	813	13.94
II	Umbriel	1.514	10.75	130	419	21.7
III	Titania	0.647	17.63	94.8	261	29.8
IV	Oberon	0.467	23.58	59.0	128	47.8
V	Miranda	—	—	—	—	—

Table VII

Satellites of Neptune

	Satellite	n	a	C	$(2C)^{3/2}$	μ/C
I	Triton	1.070r	13.33	104.6	3030	26.3
II	Nereid	—	—	—	—	—

r : retrograde

to the satellites of other planets. The computed data are shown in Tables II–VII. The situation is similar to the case of our Moon, except the eighth, the nineth, the eleventh and the twelveth satellites of Jupiter, for which the motion round Jupiter is opposite to the other satellites[2]. For these four satellites of nealy the same distance from Jupiter we have $(2C)^{3/2} < 9\mu n'$ which is the case shown in Fig. 1c and we can not ascertain the stability of their motion by such a criterion. Even from this point of view we are inclined to think

2) For Jupiter XII I have taken the data from Nicholson, Astr. Soc. Pacific, Leaflet No. 275, 1952.

that these outermost four satellites, all of retrograde motion, are quite different in their genesis from the other outer stable satellites of Jupiter VI, VII, and X of direct motion and of nearly the same distance from Jupiter. For Phoebe of Saturn and Triton of Neptune we get $(2C)^{3/2} > 9\mu n'$ and the present orbit is inside the oval of Fig. 1a, even though the motion is retrograde. It is noted that Brown has studied the motion of the eighth satellite of Jupiter but has not mentioned such difficulty in his paper[3]. For the nineth satellite Nicholson has studied[4]. The stability of the motion of a retrograde planet or satellite has been proved by Hirayama[5] by basing on the usual Laplace theory of secular perturbation.

Summary.

It is proved by the criterion of Hill for the stability of the motion of the Moon based on the Jacobi integral that the motion of the satellites in our solar system is stable under the restriction that the deviation of the actual orbit from Hill's intermediary orbit does not grow large enough to escape from the oval of zero-velocity. The motion of Phoebe and Triton is stable even though it is in retrograde motion. We can not ascertain on this criterion the stability of the outermost four satellites of Jupiter VIII, IX, XI, and XII which are in retrograde motion. Even from this circumstance we are inclined to think that these four satellites are quite different in their genesis from the outer satellites of Jupiter VI, VII and X.

3) Brown, Trans. Yale Univ. Obs., **6**, part 4, 1930 ; Brown and Brouwer, ibid., **6**, part 8, 1937.

4) Nicholson, Ap. J., **100**, 57, 1944.

5) Hirayama, Proc. Imp. Acad. Japan, **3**, 9, 1927.