120. Probability-theoretic Investigations on Inheritance. XVI$_2$. Further Discussions on Interchange of Infants.

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1. Supplementary remarks.

Similarly as remarked in § 6 of XV, the whole probability (2.15) may also be regarded as the probability of an event that, two mothers and their two children being presented, the decision is possible.

An inequality correspondiny to (6.1) *of* XV is valid here also:

$$F_0 \geqq \Psi, \quad \text{i.e.} \quad F_0 \geqq \tfrac{1}{2}F \geqq \Psi; \tag{3.1}$$

the reason being quite the same as there.

We shall now compare the probabilities derived in the preceding section with the corresponding ones, previously obtained in § 5 of XV. If the detection of interchange is possible with reference to both pairs of mother-child combinations, it is of course possible with reference to both triples of mating-child combinations. Hence, we conclude *an inequality*

$$G_0(ij) \geqq F_0(ij) \qquad (i \leqq j). \tag{3.2}$$

This inequality can also be verified directly by means of explicit expressions of its both sides. Namely, making use of (5.27) and (5.28) of XV and (2.3) and (2.4), we see that

$$\begin{aligned} G_0(ii)-F_0(ii) &= p_i^3(2(1-2S_2+S_3)-p_i+2p_i^2-p_i^3) \\ &= p_i^3\Big(2\sum_{h\neq i} p_h(1-p_h)^2+p_i(1-p_i)^2\Big) \geqq 0, \end{aligned} \tag{3.3}$$

$$\begin{aligned} G_0(ij)-F_0(ij) &= 2p_ip_j(2(1-2S_2+S_3)(p_i+p_j)-(p_i^2+p_j^2)-2p_ip_j \\ &\quad +2(p_i^3+p_j^3)-(p_i^4+p_j^4)+4p_i^2p_j^2) \\ &= 2p_ip_j\Big(2(p_i+p_j)\sum_{h\neq i,\, j} p_h(1-p_h)^2 \\ &\quad +(p_i+p_j)^2(1-p_i-p_j)^2+2p_ip_j(p_i+p_j)(1-p_i-p_j) \\ &\quad +2p_i^2p_j^2\Big) \geqq 0 \qquad (i \neq j). \end{aligned} \tag{3.4}$$

That an inequality of the same nature

$$G(ij) \geqq F(ij) \qquad (i \leqq j) \tag{3.5}$$

holds good is also a matter of course; this can also be verified in a direct manner. Hence, we see further

$$G \geqq F. \tag{3.6}$$

The general results reduce for $m=2$ to ones concerning MN

type. However, this special case can be discussed rather briefly in a direct manner. In fact, corresponding to (2.3) and (2.4), we have

(3.7) $$F_0(MM)=F_0(NN)=s^2t^2,\quad F_0(MN)=0,$$

and, corresponding to (2.5) and (2.6), we have

(3.8) $$\Psi(MM)=s^4t^2,\quad \Psi(NN)=s^2t^4,\quad \Psi(MN)=s^2t^2.$$

Thus, we get the whole probability

(3.9) $$F_{MN}=2s^2t^2(2-st).$$

Comparing this with (2.16) of XV, we see that

(3.10) $$G_{MN}-F_{MN}=2st((2-5st)(1-2st)+s^2t^2(1-st))\geqq 0.$$

By the way, the validity of (3.1) is here quite evident, since

(3.11) $$F_{0MN}-\Psi_{MN}=2s^2t^2-2s^2t^2(1-st)=2s^3t^3\geqq 0.$$

4. Illustrative examples, recessive genes being existent.

The cases where recessive genes are existent can be treated, in principle, in a similar manner, which will be illustrated by several human blood types.

In case of *ABO type*, mother-child combinations with vanishing probability are $(O;AB)$ and $(AB;O)$. Hence, we get

(4.1) $$F_0(O)=F_0(AB)=2pqr^2,\quad F_0(A)=F_0(B)=0;$$

(4.2) $$F_{0ABO}=4pqr^2.$$

It would be noticed that F_{0ABO} is identical with the whole probability C_{ABO} of (1.9) in XI of absolute non-paternity.

Making use of probabilities of mother-child combinations, we derive further, by means of an analogous process as above, the following results:

(4.3) $$\begin{aligned}\Psi(O)&=r^3\cdot(pq(p+r)+pq(q+r))=pqr^3(1+r),\\ \Psi(A)&=pr^2\cdot 2pq+pq(p+r)\cdot r^2=pqr^2(3p+r),\\ \Psi(B)&=qr^2\cdot 2pq+pq(q+r)\cdot r^2=pqr^2(3q+r),\\ \Psi(AB)&=pq(p+q)\cdot(pr^2+qr^2)=pqr^2(p+q)^2;\end{aligned}$$

(4.4) $$\Psi_{ABO}=2pqr^2(2-r+r^2).$$

Thus, the whole probability is given by the sum of (4.2) and (4.4):

(4.5) $$F_{ABO}=2pqr^2(4-r+r^2).$$

An inequality corresponding to (3.1) becomes here

(4.6) $$F_{0ABO}-\Psi_{ABO}=2pqr^2(r-r^2)\geqq 0,$$

while that corresponding to (3.6) becomes

(4.7) $$\begin{aligned}G_{ABO}-F_{ABO}=&2r^4(1-r^2)+2pq(2+2r+5r^3-r^4)\\ &-2p^2q^2(7+12r+5r^2+10r^3-r^4)\\ &+4p^3q^3(1-2r-4r^2)-2p^4q^4\geqq 0,\end{aligned}$$

as is readily seen in view of $pq\leqq(1-r)^2/4$.

In case of A_1A_2BO *type*, the results are as follows:

$$
(4.8)\quad \begin{aligned}
&F_0(O)=2pqr^2, \quad F_0(A_1)=0, \quad F_0(A_2)=2p_1p_2q(p_2+2r),\\
&F_0(B)=0, \quad F_0(A_1B)=2p_1qr^2+2p_1p_2q(p^2+2r),\\
&\qquad F_0(A_2B)=2p_2qr^2;
\end{aligned}
$$

$$
(4.9)\quad F_{0A_1A_2BO}=4pqr^2+4p_1p_2q(p_2+2r)=F_{0ABO}+4p_1p_2q(p_2+2r);
$$

the last quantity coincides also with $C_{A_1A_2BO}$ in (1.7) of XI.

$$
(4.10)\quad \begin{aligned}
&\Psi(O)=r^3\cdot pq(1+r)+p_2r^2\cdot p_1q(1+r),\\
&\Psi(A_1)=p_1r^2\cdot 2pq+p_1p_2(p_2+2r)\cdot 2p_1q+p_1q(p+r)\cdot(p_2+r)^2\\
&\qquad +p_1p_2q\cdot r^2,\\
&\Psi(A_2)=p_2r^2\cdot(2pq-p_1q(p+q))+p_2(p_2^2+3p_2r+r^2)\cdot p_1q(2-p_1-q)\\
&\qquad +p_2q(p_2+r)\cdot r^2,\\
&\Psi(B)=qr^2\cdot 2pq+p_2qr\cdot 2p_1q+p_1q(q+r)\cdot(p_2+r)^2+p_2q(q+r)\cdot r^2,\\
&\Psi(A_1B)=p_1q(p_1+q)\cdot(p_1r^2+qr^2+p_1p_2(p_2+2r)+p_2qr+p_2q(p_2+r))\\
&\qquad +p_1p_2q\cdot(p_1+q)r^2,\\
&\Psi(A_2B)=p_2q(p_2+r)\cdot 2p_1q+p_1p_2q\cdot(r^2-r^3+p_2(p_2+2r-r^2))\\
&\qquad +p_2q(p_2+q)\cdot(r^2-r^3).
\end{aligned}
$$

Summing up and comparing with (4.4), we get

$$
(4.11)\quad \begin{aligned}
\Psi_{A_1A_2BO}=\Psi_{ABO}+2p_1p_2q\{&p_2(1+p_1+q+r)+r(1+3p_1+3q)\\
&+p_2^3+4p_2^2r+4p_2r^2+3r^3\} \qquad (p_1+p_2=p).
\end{aligned}
$$

The whole probability is given by

$$
(4.12)\quad \begin{aligned}
F_{A_1A_2BO}=F_{ABO}+2p_1p_2q\{&p_2(3+p_1+q+r)+r(5+3p_1+3q)\\
&+p_2^3+4p_2^2r+4p_2r^2+3r^3\} \qquad (p_1+p_2=p).
\end{aligned}
$$

In cases of Q and $Qq_{\pm}$ *types* there are no mother-child combinations with vanishing probability. Hence, in coincidence with (1.10) of XI, we get

$$
(4.13)\quad F_{0Q}=F_{0Qq\pm}=0.
$$

Based on the same reason or on an analogous inequality as (3.1), we get further

$$
(4.14)\quad \Psi_Q=\Psi_{Qq\pm}=0,
$$

whence it follows that

$$
(4.15)\quad F_Q=F_{Qq\pm}=0.
$$

Namely, Q as well as $Qq_{\pm}$ *types* have no effect upon the detection of interchange of infants with only reference to mother-child combinations.

It is noticed that the discontinuity between ABO and MN types appears here also. In fact,

$$
(4.16)\quad [F_{ABO}]^{r=0}-[F_{MN}]^{(s,t)=(p,q)}=-2p^2q^2(2-pq).
$$

But, there is no discontinuity between ABO and Q types, A_1A_2BO and ABO as well as $Qq_{\pm}$ and Q types.

5. Maximizing distributions.

The distribution of genes maximizing the respective probability can be determined in a usual way.

We first consider the probability (3.9) in case of *MN type*, for which we get

$$dF_{MN}/d(st)=2st(4-3st)>0 \qquad (0<st<1/4). \tag{5.1}$$

Hence, the maximizing distribution is given by

$$s=t=1/4; \quad \overline{M}=\overline{N}=1/4, \quad \overline{MN}=1/2, \tag{5.2}$$

yielding the maximum value

$$(F_{MN})^{\max}=7/32=0.21875. \tag{5.3}$$

The whole probability (2.17) attains its stationary value

$$(F)^{\text{stat}}=\left(1-\frac{1}{m}\right)\left(1-\frac{8}{m^2}+\frac{14}{m^3}-\frac{5}{m^4}\right) \tag{5.4}$$

for the symmetric distribution

$$p_i=1/m \qquad (i=1,\cdots,m), \tag{5.5}$$

which will perhaps be the actual maximizing one. The value (5.4) increases with m and tends to 1 as $m\to\infty$. In fact,

$$\frac{d}{d(1/m)}(F)^{\text{stat}}=-\left(1-\frac{2}{m}\right)\left(1+\frac{3}{m}+\frac{15}{m}\left(1-\frac{2}{m}\right)+\frac{16}{m^3}\right)-\frac{7}{m^4}<0 \qquad (m\geqq 2).$$

In case of *ABO type*, the probability (4.5) can be regarded as a function of two independent variables r and pq with admissible range $0\leqq r\leqq 1$, $0\leqq pq\leqq (1-r)^2/4$. Since it is linear in pq, the maximum will be attained when $pq=(1-r)^2/4$, that is

$$p=q=(1-r)/2, \qquad 0<r<1. \tag{5.6}$$

Substituting this into (4.5), F_{ABO} can be regarded as a function of r alone which will be denoted by $f(r)$. The equation

$$0=2f'(r)=r(1-r)(1-2r)(8-3r+3r^2) \tag{5.7}$$

possesses a unique root, namely 1/2, contained in $0<r<1$. Thus, together with (5.6), the maximizing distribution is given by

$$\begin{aligned} &r=1/2, \qquad p=q=1/4; \\ &\overline{O}=1/4, \qquad \overline{A}=\overline{B}=5/16, \qquad \overline{AB}=1/8; \end{aligned} \tag{5.8}$$

the maximum value being

$$(F_{ABQ})^{\max}=15/128=0.1172. \tag{5.9}$$

The case of A_1A_2BO *type*, though somewhat troublesome, can be treated in a similar way. The cases Q and $Qq_\pm$ *types* having vaninshing probability, the maximum problem is non-sense.

—To be continued—