# 116. *Probability-theoretic Investigations on Inheritance. XV$_2$. Detection of Interchange of Infants.*

By Yûsaku KOMATU.

Department of Mathematics, Tokyo Institute of Technology and Department of Legal Medicine, Tokyo Medical and Dental University.

(Comm. by T. FURUHATA, M.J.A., Nov. 12, 1952.)

## 3. Another method of attack.

Wiener has arrived at his result on the whole probability of detecting interchange of infants, given in (2.16), by calculating the 36 partial probabilities one after another and then summing up them. But, this method will become rapidly troublesome as the number of possible types of an inherited character increases. In fact, if there exist, in general, $m^*$ different phenotypes, then the possible matings amount to $m^*(m^*+1)/2$ kinds so that the combinations of matings to be considered amount to $m^{*2}(m^*+1)^2/4$ in number. For instance, this number is equal to 100 or 441 for $m^*=4$ ($ABO$ blood type) or $m^*=6$ ($A_1A_2BO$ blood type), respectively. Even when the combinations with identically vanishing probability are omitted and further only the half of the remaining combinations are considered in view of symmetry, they amount yet to

$$(3.1)\quad \tfrac{1}{2}(\tfrac{1}{4}m^{*2}(m^*+1)^2-\tfrac{1}{2}m^*(m^*+1))=\tfrac{1}{8}(m^*-1)m^*(m^*+1)(m^*+2),$$

which is equal to 45 or 210 for $m^*=4$ or $m^*=6$, respectively.

On account of the reason just stated, we shall now again deal with the problem by *another method* of attack which will directly apply also to general mode of an inherited character.

First, we observe the first mating of $M\times M$. Then, the detection of interchange is possible when and only when the true child of the second mating is not $M$. Since the child different from $M$ appears with frequency $1-s^2$, we get the partial probability

$$(3.2)\qquad s^4(1-s^2).$$

Similarly, in case of the first mating $N\times N$, the partial probability is given by

$$(3.3)\qquad t^4(1-t^2),$$

and in case of the first mating $M\times N$, it is equal to

$$(3.4)\qquad 2s^2t^2(1-2st).$$

The above three are the cases where the first mating can produce only one type of child, Next, we observe the first mating of $M\times MN$, so that the child $N$ cannot be produced. Hence, if the apparent child of the first mating is $N$, then the interchange is detectable. But, besides this, there are further cases where the detection of interchange is possible; namely, the case where the apparent child of the second mating is $M$ and the second mating can produce $MN$ but cannot $M$——the apparent child of the first mating is then necessarily $MN$——, and the case where the apparent child of the second mating is $MN$ and the second mating can produce $M$ but cannot $MN$——the apparent child of the first mating is then necessarily $M$——. The former corresponds to the second mating of $M\times N$ or $N\times MN$ and the latter to that of $M\times M$. But, among those, the case where the second mating $N\times MN$ produces $N$ having been already taken into account, there remains, in $N\times MN$, to consider the true child $MN$ which appears with probability 1/2. Thus, the partial probability with respect to the first mating of $M\times MN$ becomes

$$(3.5)\quad 4s^3t\cdot t^2+2s^3t(2s^2t^2+4st^3\cdot\tfrac{1}{2})+2s^3t\cdot s^4=2s^3t(s^4+2t^2(1+s)).$$

Similarly, in case of the first mating $N\times MN$, the partial probability is equal to

$$(3.6)\quad 4st^3\cdot s^2+2st^3(2s^2t^2+4s^3t\cdot\tfrac{1}{2})+2st^3\cdot t^4=2st^3(t^4+2s^2(1+t)).$$

Last, the first mating $MN\times MN$ can produce every type of child. But, if the apparent child of the second mating is $M$; $N$; or $MN$ and the second mating cannot produce the respective type, then the detection of interchange is possible. Such matings are $M\times N$, $N\times N$, $N\times MN$; $M\times M$, $M\times N$, $M\times MN$; or $M\times M$, $N\times N$, respectively. Hence, we get the partial probability

$$(3.7)\quad \begin{aligned} &s^2t^2(2s^2t^2+t^4+4st^3)+s^2t^2(s^4+2s^2t^2+4s^3t)+2s^2t^2(s^4+t^4)\\ &\qquad\qquad =s^2t^2(3-8st+2s^2t^2).\end{aligned}$$

The partial probabilities obtained in (3.2) to (3.7) are the ones already listed in the last column of the table at the end of §2. It is the matter of course that the total sum of these partial probabilities is just equal to the *whole probability* given in (2.16).

We notice here that the process of the method just mentioned can be performed in an automatical manner. To explain it, we now prepare a table corresponding to the one listed in §3 of I for general mode of inheritance.

We first observe the mating $M\times M$ accompanied by its child $M$. The row of $M\times M$ in the table contains the vanishing probability for children $N$ and $MN$. Accordingly, we mark the probabilities lying in the same columns as $N$ and $MN$; indicated by the mark ▲ in the table. The sum of the marked probabilities is equal to $1-s^2$, yielding the second factor of (3.2). To each mating we apply an analogous procedure, obtaining for $M\times N$, $M\times MN$, $N\times N$, $N\times MN$, $MN\times MN$ the quantities $1-2st$, $t^2$, $1-t^2$, $s^2$, 0, respectively, among which the first four yield the second factor of (3.4), that of the first term in (3.5), that of (3.3), and that of the first term in (3.6), respectively. The total sum of these probabilities multiplied by the corresponding mating-probabilities represents the probability of detecting the interchange of infants by means of the first mating and an interchanged infant alone.

| Mating | Prob. of mating | Freq. of child | | |
|---|---|---|---|---|
| | | $M$ | $N$ | $MN$ |
| $M\times M$ | $s^4$ | $s^4$ | 0 | 0 |
| $M\times N$ | $2s^2t^2$ | 0 | 0 | $2s^2t^2$▲● |
| $M\times MN$ | $4s^3t$ | $2s^3t$ | 0 | $2s^3t$▲ |
| $N\times N$ | $t^4$ | 0 | $t^4$▲ | 0 |
| $N\times MN$ | $4st^3$ | 0 | $2st^3$▲ | $2st^3$▲● |
| $MN\times MN$ | $4s^2t^2$ | $s^2t^2$ | $s^2t^2$▲ | $2s^2t^2$▲ |

The matings $M\times M$, $M\times N$ and $N\times N$ can produce only one kind of respective child, while the remaining three matings can produce at least two kinds of child. We observe the mating $M\times MN$ which can never produce $N$, as shown by vanishing probability in the table. In the column of $M$ the vanishing probability appears for $M\times N$, $N\times N$ and $N\times MN$. In the rows of three last-mentioned matings the non-vanishing probabilities except those marked already in the first process with respect to $M\times MN$ are marked now; indicated by the mark ●. The sum of the marked probabilities is equal to $2s^2t^2+2st^3$, yielding the second factor of the second term in the left-hand side of (3.5). To the column of $MN$ an analogous process is applied, yielding the second factor of the third term, i.e., $s^4$, in the left-hand side of (3.5). To each apparent child of the remaining second matings $N\times N$ and $N\times MN$ which is possible from first mating an analogous process will be applied. We then obtain the second factors of the second and third terms in the left-hand side of (3.6), as well as those of the first, second and third terms in the left-hand side of (3.7). The total sum of these probabilities multiplied by the corresponding frequencies of apparent children represents the probability of detecting the interchange only by taking the second triple into account.

The procedure just explained with respect to $MN$ type can evidently be generalized also to the general mode of inheritance. The details will be discussed in the following sections.

**4. Preparatory considerations.**

In order to prevent the interruption on the way of our main discourse, we make here beforehand some preparatory considerations. Let us designate by

$$(4.1)\qquad \varphi(-ij,+hk)\qquad (i,j,h,k=1,\ldots,m;\,(ij)\neq(hk))$$

*the probability of an event that a mating not able to produce* $A_{ij}$ *appears and is accompanied by its child* $A_{hk}$. Consequently, for instance, the probability of an event that a mating not able to produce $A_{ij}$ appears and is accompanied by its child $A_{hk}$ or $A_{fg}$ $((fg)\neq(hk))$ is given additively by

$$(4.2)\qquad \varphi(-ij,+hk+fg)=\varphi(-ij,+hk)+\varphi(-ij,+fg).$$

We shall now determine the quantities defined in (4.1) explicitly.

First, for $\varphi(-ii,+ih)$ $(h\neq i)$, the following matings, the order being indifferent, are to be considered:

$$(4.3)\qquad A_{ii}\times A_{hh},\ A_{ii}\times A_{jh},\ A_{ih}\times A_{hh},\ A_{ih}\times A_{jh},\ A_{ij}\times A_{hh},\ A_{il}\times A_{jh}\qquad (j,l\neq i,h).$$

These matings accompanied by a child $A_{ih}$ appear with respective probabilities

$$(4.4)\qquad 2p_i^2p_h^2,\ 2p_i^2p_hp_j,\ 2p_ip_h^3,\ 2p_ip_h^2p_j,\ 2p_ip_h^2p_j,\ 2p_ip_hp_jp_l.$$

The sum of all the probabilities contained in (4.4) gives

$$(4.5)\qquad \begin{aligned}\varphi(-ii,+ih)&=2p_ip_h\Big(p_ip_h+p_h^2+(p_i+2p_h)\sum_{j\neq i,h}p_j+\sum_{j,l\neq i,h}p_jp_l\Big)\\&=2p_ip_h(1-p_i)\qquad (h\neq i).\end{aligned}$$

Next, for $\varphi(-ih,+ii)$ $(h\neq i)$, the matings to be considered are

$$(4.6)\qquad A_{ii}\times A_{ii},\ A_{ii}\times A_{ij},\ A_{ij}\times A_{ij},\ A_{ij}\times A_{il}\qquad (j,l\neq i,h;\,j\neq l),$$

with corresponding probabilities of producing $A_{ii}$:

$$(4.7)\qquad p_i^4,\ 2p_i^3p_j,\ p_i^2p_j^2,\ 2p_i^2p_jp_l.$$

Thus, we get by summation

$$(4.8)\qquad \begin{aligned}\varphi(-ih,+ii)&=p_i^2\Big(p_i^2+2p_i\sum_{j\neq i,h}p_j+\sum_{j\neq i,h}p_j^2+2\sum_{j,l\neq i,h}{}'p_jp_l\Big)\\&=p_i^2(1-p_h)^2\qquad (h\neq i).\end{aligned}$$

For $\varphi(-ii,+hh)$ $(h\neq i)$, the matings to be considered are

$$(4.9)\qquad A_{hh}\times A_{hh},\ A_{hh}\times A_{ih},\ A_{hh}\times A_{hj},\ A_{ih}\times A_{hj},\ A_{hj}\times A_{hj},\ A_{hj}\times A_{hl}\qquad (j,l\neq i,h;\ j\neq l),$$

with corresponding probabilities of producing $A_{hh}$:

$$(4.10)\qquad p_h^4,\ 2p_ip_h^3,\ 2p_h^3p_j,\ 2p_ip_h^2p_j,\ p_h^2p_j^2,\ 2p_h^2p_jp_l.$$

Thus, we get

$$(4.11)\quad \varphi(-ii,+hh)=p_h^2\Big(p_h^2+2p_ip_h+2(p_h+p_i)\sum_{j\neq i,h}p_j+\sum_{j\neq i,h}p_j^2 +2\sum_{j,l\neq i,h}{}' p_jp_l\Big)=p_h^2(1-p_i^2)\qquad (h\neq i).$$

For $\varphi(-ii,+hk)\,(h,k\neq i;\, h\neq k)$, the matings to be considered are

$$(4.12)\quad \begin{gathered} A_{hh}\times A_{hk},\ A_{kk}\times A_{hk},\ A_{hh}\times A_{kk},\ A_{hk}\times A_{hk},\ A_{hh}\times A_{ik},\ A_{ih}\times A_{hk},\\ A_{kk}\times A_{ih},\ A_{ik}\times A_{hk},\ A_{ih}\times A_{kj},\ A_{ik}\times A_{hj},\ A_{hh}\times A_{kj},\ A_{hk}\times A_{hj},\\ A_{kk}\times A_{hj},\ A_{hk}\times A_{kj},\ A_{hj}\times A_{kj},\ A_{hj}\times A_{kl},\ A_{kj}\times A_{hl}\\ (j,\, l\neq i,\, h,\, k;\, j\neq l),\end{gathered}$$

with corresponding probabilities of producing $A_{hk}$:

$$(4.13)\quad \begin{gathered} 2p_h^3p_k,\ 2p_hp_k^3,\ 2p_h^2p_k^2,\ 2p_h^2p_k^2,\ 2p_ip_h^2p_k,\ 2p_ip_h^2p_k,\\ 2p_ip_hp_k^2,\ 2p_ip_hp_k^2,\ 2p_ip_hp_kp_j,\ 2p_ip_hp_kp_j,\ 2p_i^2p_h^2p_k,\ 2p_h^2p_kp_j,\\ 2p_hp_k^2p_j,\ 2p_hp_k^2p_j,\ 2p_hp_kp_j^2,\ 2p_hp_kp_jp_l,\ 2p_hp_kp_jp_l.\end{gathered}$$

Thus, we get by summation

$$(4.14)\quad \begin{aligned}\varphi(-ii,+hk)&=2p_hp_k\Big(p_h^2+p_k^2+2p_hp_k+2p_i(p_h+p_k)\\ &\qquad +2(p_i+p_h+p_k)\sum_{j\neq i,h,k}p_j+\sum_{j\neq i,h,k}p_j^2+2\sum_{j,l\neq i,h,k}{}'p_jp_l\Big)\\ &=2p_hp_k(1-p_i^2)\qquad (h,\,k\neq i;\,h\neq k).\end{aligned}$$

For $\varphi(-hk,+ii)\,(h,\,k\neq i;\,h\neq k)$, the matings to be considered are

$$(4.15)\quad \begin{gathered} A_{ii}\times A_{ii},\ A_{ii}\times A_{ih},\ A_{ii}\times A_{ik},\ A_{ih}\times A_{ih},\ A_{ik}\times A_{ik},\ A_{ii}\times A_{ij},\\ A_{ih}\times A_{ij},\ A_{ik}\times A_{ij},\ A_{ij}\times A_{ij},\ A_{ij}\times A_{il}\quad (j,\,l\neq i,\,h,\,k;\,j\neq l),\end{gathered}$$

with corresponding probabilities of producing $A_{ii}$:

$$(4.16)\quad \begin{gathered} p_i^4,\ 2p_i^3p_h,\ 2p_i^3p_k,\ p_i^2p_h^2,\ p_i^2p_k^2,\ 2p_i^3p_j,\\ 2p_i^2p_hp_j,\ 2p_i^2p_kp_j,\ p_i^2p_j^2,\ 2p_i^2p_jp_l.\end{gathered}$$

Thus, we get

$$(4.17)\quad \begin{aligned}\varphi(-hk,+ii)&=p_i^2\Big(p_i^2+2p_i(p_h+p_k)+p_h^2+p_k^2\\ &\qquad +2(p_i+p_h+p_k)\sum_{j\neq i,h,k}p_j+\sum_{j\neq i,h,k}p_j^2+2\sum_{j,l\neq i,h,k}{}'p_jp_l\Big)\\ &=p_i^2(1-2p_hp_k)\qquad (h,\,k\neq i;\,h\neq k).\end{aligned}$$

For $\varphi(-ih,+ik)\,(h,\,k\neq i;\,h\neq k)$, the matings to be considered are

$$(4.18)\quad \begin{gathered} A_{ii}\times A_{ik},\ A_{kk}\times A_{ik},\ A_{ik}\times A_{ik},\ A_{ii}\times A_{kk},\ A_{ih}\times A_{kk},\ A_{ik}\times A_{ij},\\ A_{ii}\times A_{kj},\ A_{ih}\times A_{kj},\ A_{ik}\times A_{kj},\ A_{kk}\times A_{ij},\ A_{ij}\times A_{kl}\ (j,l\neq i,h,k),\end{gathered}$$

with corresponding probabilities of producing $A_{ik}$:

$$(4.19)\quad \begin{gathered} 2p_i^3p_k,\ 2p_ip_k^3,\ 2p_i^2p_k^2,\ 2p_i^2p_k^2,\ 2p_ip_hp_k^2,\ 2p_i^2p_kp_j,\\ 2p_i^2p_kp_j,\ 2p_ip_hp_kp_j,\ 2p_ip_k^2p_j,\ 2p_ip_k^2p_j,\ 2p_ip_kp_jp_l.\end{gathered}$$

Thus, we get

$$(4.20)\quad \begin{aligned}\varphi(-ih,+ik)&=2p_ip_k\Big(p_i^2+p_k^2+(2p_i+p_h)p_k\\ &\qquad +(2p_i+p_h+2p_k)\sum_{j\neq i,h,k}p_j+\sum_{j,l\neq i,h,k}p_jp_l\Big)\\ &=2p_ip_k(1-p_h-p_ip_h)\qquad (h,\,k\neq i;\,h\neq k).\end{aligned}$$

Last, for $\varphi(-ij,+hk)\ (i\neq j;\ h\neq k;\ i, j\neq h, k)$, the matings to be considered are

$$
(4.21)\quad \begin{aligned}
&A_{hh}\times A_{hk},\ A_{kk}\times A_{hk},\ A_{hh}\times A_{kk},\ A_{hk}\times A_{hk},\ A_{ih}\times A_{ik},\ A_{jh}\times A_{jk},\\
&A_{ih}\times A_{hk},\ A_{hh}\times A_{ik},\ A_{jh}\times A_{hk},\ A_{hh}\times A_{jk},\ A_{ik}\times A_{hk},\ A_{kk}\times A_{ih},\\
&A_{jk}\times A_{hk},\ A_{kk}\times A_{jh},\ A_{hh}\times A_{kl},\ A_{hk}\times A_{hl},\ A_{kk}\times A_{hl},\ A_{hk}\times A_{kl},\\
&A_{ih}\times A_{kl},\ A_{ik}\times A_{hl},\ A_{jh}\times A_{kl},\ A_{jk}\times A_{hl},\ A_{hl}\times A_{kf}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (l, f\neq i, j, h, k),
\end{aligned}
$$

with corresponding probabilities of producing $A_{hk}$:

$$
(4.22)\quad \begin{aligned}
&2p_h^3p_k,\ 2p_hp_k^3,\ 2p_h^2p_k^2,\ 2p_h^2p_k^2,\ 2p_i^2p_hp_k,\ 2p_j^2p_hp_k,\\
&2p_ip_h^2p_k,\ 2p_ip_h^2p_k,\ 2p_jp_h^2p_k,\ 2p_jp_h^2p_k,\ 2p_ip_hp_k^2,\ 2p_ip_hp_k^2,\\
&2p_jp_hp_k^2,\ 2p_jp_hp_k^2,\ 2p_h^2p_kp_l,\ 2p_h^2p_kp_l,\ 2p_hp_k^2p_l,\ 2p_hp_k^2p_l,\\
&2p_ip_hp_kp_l,\ 2p_ip_hp_kp_l,\ 2p_jp_hp_kp_l,\ 2p_jp_hp_kp_l,\ 2p_hp_kp_lp_f.
\end{aligned}
$$

Thus, we get by summation

$$
(4.23)\quad \begin{aligned}
\varphi(-ij,+hk) &= 2p_hp_k\Big(p_h^2+p_k^2+2p_hp_k+p_i^2+p_j^2+2(p_i+p_j)(p_h+p_k)\\
&\qquad +2(p_h+p_k+p_i+p_j)\sum_{l\neq i, j, h, k} p_l+\sum_{l, f\neq i, j, h, k} p_lp_f\Big)\\
&= 2p_hp_k(1-2p_ip_j) \qquad (i\neq j;\ h\neq k;\ i, j\neq h, k).
\end{aligned}
$$

All the possible cases have thus been essentially worked out. It would be noticed that if $A_{ij}$ and $A_{hk}$ have no common gene, then the relation, also immediately evident,

$$
(4.24)\quad \varphi(-ij,+hk)=\bar{A}_{hk}(1-\bar{A}_{ij})
$$

holds, as really seen in (4.11), (4.14), (4.17) and (4.23), while if $A_{ij}$ and $A_{hk}$ have a common gene, the same is not true, as seen in (4.5), (4.8) or (4.20).

*—To be continued—*