71. *Topology of Standard Path Spaces and Homotopy Theory. I*

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This is the first of a series of notes, whose aim is to clarify the homological structure of the path space $\Omega(X, A)=\{f : I^{1)}\to X \mid f(0)=*, f(1)\in A\}$ by means of "*standard path space*" and to investigate the homotopical structure of spaces. The paper of J-P. Serre[2)] based on the singular homology theory of fibre spaces shows how the loop space $\Omega(X)=\Omega(X, *)$ is applied to the calculation of the Hurewicz homotopy groups $\pi_p(X)$ of X.

It was proved by J. B. Giever[3)] that to every space X there exists a CW-complex $P(X)$ and a map of $P(X)$ into X inducing isomorphisms of the homotopy groups of $P(X)$ onto those of X. A problem to determine the homological structures of $P(\Omega(X))$ from those of $P(X)$ is closely related with Serre's theory. For the simply connected space X, this problem can be solved by selecting complexes $K(X)$ and $\omega(K(X))$, so-called a standard complex and a standard path complex respectively, when the complex $\omega(K(X))$ is combinatorially constructed from $K(X)$.

Here we give definitions of standard spaces and standard paths in them. The set of standard paths in a standard complex K, whose end points are in a subcomplex L of K, forms a closed subset $\omega(K, L)$ of $\Omega(K, L)$. The standard path space $\omega(K, L)$ is a CW-complex and is constructed from K and L by a combinatiorial method.

The fundamental result in this note is roughly stated as follows; *the injection : $\omega(K, L)\to\Omega(K, L)$ induces isomorphisms of homotopy and homology groups of $\omega(K, L)$ onto those of $\Omega(K, L)$.*

Our theory is applied to determine the orders of homotopy groups $\pi_p(S^n)$ of n-sphere S^n for $p\leqq n+8$.

§ 1. **Standard Paths in a Suspended Space.** Let $E(X)$ be a suspended space of a space X, which is obtained from $X\times I$ by shrinking a subset $*\times I\cup X\times \dot{I}$[4)] to a single point $*$, and let d: $X\times I\to E(X)$ be its shrinking map. Assume that a real function ρ of X is given such that ρ is positive excepting $\rho(*)=0$. Then define a standard path $l(x_1, \dots, x_n;\ y, t)$: $I\to E(X)$ by a formula

$$(A)\quad l(x_1, \dots, x_n;\ y, t)\ (s)=\begin{cases} d(x_i,\ (s-s_{i-1})/\rho(x_i)) & s_{i-1}\leqq s\leqq s_i, \\ d(y,\ (s-s_n)/\rho(y)) & s_n\leqq s\leqq 1, \end{cases}$$

where $x_i\in X$, $y\in A\subset x$, $t\in I$, $s_0=0$ and $s_i=\sum_{k=1}^{i}\rho(x_k)/(\sum_{k=1}^{n}\rho(x_k)+t\cdot\rho(y))$

for $i=1, \ldots, n$. The path $l(x_1, \ldots, x_n; y, t)$ starts at the base point $*$ and ends at a point $d(y, t)$ of $E(A)=d(A\times I)$. The set of paths $l(x_1, \ldots, x_n; y, t)$ forms a closed subset $\omega(E(X), E(A))$ of the path space $\Omega(E(X), E(A))$, called a standard path space of the pair $(E(X), E(A))$. Denote a loop $l(x_1, \ldots, x_n; *, 0)$ by $l(x_1, \ldots, x_n)$ and denote the set $\omega(E(X), *)$ of standard loops by $\omega(E(X))$. Let $(X)^n$ be an n-fold product of X, identify $(X)^{n-1}$ to $(X)^{n-1}\times * \subset (X)^n$ and set $\bigcup_n (X)^n=(X)^\infty$. Then the space $\omega(E(X), E(A))$ is obtained from $(X)^\infty \times A\times I$ by the following identifications:

$(x_1, \ldots, x_n; y, 0)\equiv(x_1, \ldots, x_{n-1}; x_n, 1)\equiv(x_1, \ldots, x_n; *, t)$ and $(x_1, \ldots, x_n; y, t)\equiv(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n; y, t)$ if $x_i=*$.

Specially, the space X is naturally imbedded into $\omega(E(X), E(A))$ by a correspondence: $x\to l(x)$.

A pair (X, A) is said to have *a homotopy extension property* if $X\times(0)\cup A\times I$ is a deformation retract of $X\times I$. Then our primary result is;

Theorem I. *If (X, A) and $(A, *)$ have the homotopy extension property and if X is arcwise connected and (X, A) is 1-connected, then the injection homomorphisms of homotopy groups $i_*: \pi_p(\omega(E(X)))\to \pi_p(\Omega(E(X)))$ and $i_*: \pi_p(\omega(E(X), E(A)))\to\pi_p(\Omega(E(X), E(A)))$ are all isomorphisms*[5].

For a map $f: (Y, y_*)\to(\omega(E(X)), l(*))$ we define a suspension $Ef: E(Y)\to E(X)$ of X by setting $Ef(d(y, t))=f(y)(t)$. If Y is a finite polyhedron, the homotopy classes of f and Ef correspond one-to-one. Since the set of all classes of Ef coincides to the fundamental group of a function space $E(X)_0^Y=\{g: Y\to E(X) \mid g(y_*)=*\}$, the homotopy classes of f form a group. This group is a generalization of the cohomotopy groups of E. Spanier[6]. We mention the fact that there are suspension isomorphisms

$$\pi_p(\omega(E(X))\approx\pi_{p+1}(E(X)),$$
$$\pi_p(\omega(E(X), E(A))\approx\pi_{p+1}(E(X), E(A)),$$
and
$$\pi_p(\omega(E(X)), X)\approx\pi_{p+1}(E(X); \hat{X}_+, \hat{X}_-),$$

where $X=d\left(X\times\left[\frac{1}{2}, 1\right]\right)$, $\hat{X}_-=d\left(X\times\left[0, \frac{1}{2}\right]\right)$ and $\pi_{p+1}(E(X); \hat{X}_+, \hat{X}_-)$ is the homotopy groups of triad[7].

§2. **Definition of Standard Paths.** In this section we define a standard complex ${}^mK=E(K_0; f_1, E_1; \ldots; f_m, E_m)$ and standard paths in K inductively. For $n=0$, ${}^0K$ is the suspended space $E(K_0)$ of a CW-complex K_0 and the standard path in it was already defined in §1 by the formula (A) with respect to the function ρ of K_0. Define a real function ρ_0 of $\omega({}^0K)$ by $\rho_0(l(x_1, \ldots, x_n))=\rho(x_1)+\cdots+\rho(x_n)$. Suppose the standard complex ${}^{m-1}K$, the space $\omega({}^{m-1}K)$ of the standard loops in ${}^{m-1}K$ and a real function ρ_{m-1} of $\omega({}^{m-1}K)$ are already defined such that ${}^{m-1}K$ is a CW-complex and ρ_{m-1} is positive excepting $\rho_{m-1}(l(*))=0$. Let f_m be a map of (S, s_*) into $(\omega({}^{m-1}K), l(*))$, where $S=\bigvee_\alpha S_\alpha^{n_\alpha}$ is the sum of n_α-spheres $S_\alpha^{n_\alpha}$ $(n_\alpha \geqq 1)$ having a single point s_* in common. Define a map F_m: $E(S) \to {}^{m-1}K$ by setting $F_m(d(y, t))=f_m(y)(t)$, where $E(S)=\bigvee_\alpha S_\alpha^{n_\alpha+1}$ is the suspended space of S and d: $S \times I \to E(S)$ is its shrinking map. Then the standard complex ${}^mK={}^{m-1}K \cup \bar{\varepsilon}_m=(\bar{\varepsilon}_m \bigvee_\alpha \varepsilon_\alpha^{n_\alpha+2})$ is obtained from K attaching the $(n_\alpha+2)$-cells $\varepsilon_\alpha^{n_\alpha+2}$ by $F_m | S_\alpha^{n_\alpha+1}$. Let $\omega({}^{m-1}K) \cup \varepsilon_m(\varepsilon_m=\bigvee_\alpha \varepsilon_\alpha^{n_\alpha+1})$ be a complex obtained from $\omega({}^{m-1}K)$ attaching the $(n_\alpha+1)$-cells $\varepsilon_\alpha^{n_\alpha+1}$ by $f_m | s_\alpha^{n_\alpha}$, then there exists a map

$$d_m: \ (\omega({}^{m-1}K) \cup \varepsilon_m) \times I \to {}^mK$$

such that $d_m(l, t)=l(t)$ for $l \in \omega({}^{m-1}K)$, $d_m((\omega({}^{m-1}K) \cup \varepsilon_m) \times \dot{I})=*$ and d_m is homeomorphic elsewhere. The function ρ_{m-1} of $\omega({}^{m-1}K)$ is extendable to whole of ε_m positively excepting $\rho_{m-1}(l(*))=0$. A standard path

$$l(x_1, \ldots, x_n; y, t): \ I \to {}^mK$$

is defined for x_i, $y \in \omega({}^{m-1}K) \cup \varepsilon_m$ and $t \in I$ by the formula (A), replacing the operations d and ρ by d_m and ρ_{m-1} respectively. The space of all standard loops $l(x_1, \ldots, x_n)=l(x_1, \ldots, x_n; *, 0)$ will be denoted by $\omega({}^mK) \subset \Omega({}^mK)$. Finally we define a real function ρ_m of $\omega({}^mK)$ by $\rho_m(l(x_1, \ldots, x_n))=\sum_{i=1}^{n} \rho_{m-1}(x_i)$.

In general, standard complex $K=E(K_0; f_1, \varepsilon_1; \ldots; f_m, \varepsilon_m; \ldots)$ is a CW-complex defined by $K=\bigvee_m {}^mK$. Since a correspondence $x \to l(x)$ $(x \in \omega({}^{m-1}K))$ is an imbedding of $\omega({}^{m-1}K)$ into $\omega({}^mK)$, we may define a space $\omega(K)$ of standard loops in K by $\omega(K)=\bigvee_m \omega({}^mK)$. Let L be a subcomplex of K, then L is represented by a form $E(L_0; f_1', \varepsilon_1'; \ldots; f_m', \varepsilon_m'; \ldots)$ where $\varepsilon_i'=\varepsilon_i \frown L$ and $f_i'=f_i | \varepsilon_i'$. The set of all paths $l(x_1, \ldots, x_n; y, t)$, for $x_i \in \omega({}^{m-1}K) \cup \varepsilon_m$, $y \in \omega({}^{m-1}L) \cup \varepsilon_m'$ and $t \in I$, forms a colsed subset $\omega({}^mK, {}^mL)$ of $\Omega({}^mK, {}^mL)$ where ${}^mL=L \frown {}^mK$. Let us define a space of standard paths $\omega(K, L)$ by $\bigvee_m \omega({}^mK, {}^mL)$. Then the space $\omega(K, L)$ is constructed from K and L combinatorially as follows, and this space becomes a CW-complex. Define a continuous map

$$\vee: \ \omega(K) \times \omega(K, L) \to \omega(K, L)$$

by $\vee(l(x)), l'(x', t))=l \vee l'(x; x', t)$ for paths x, x' of ${}^mK$ (m; sufficiently large). Then $\vee(\omega({}^mK) \times \omega({}^mK))=\omega({}^mK)$, $l \vee l(*)=l(*) \vee l=l$ and $l \vee (l' \vee l'')=(l \vee l') \vee l''$, and this implies the simplicity of $\omega(K)$ and $(\Omega(K), \omega(K))$. Considering a correspondence: $(x_1, \ldots, x_n; y, t) \to l(x_1, \ldots, x_n; y, t)$, we have that the standard path space $\omega({}^mK, {}^mL)$ is constructed from $(\omega({}^{m-1}K) \cup \varepsilon_m)^\infty \times (\omega({}^{m-1}L) \cup \varepsilon_m') \times I$ by the following identifications;

$$(x_1, \ldots, x_n; y, 0) \equiv (x_1, \ldots, x_{n-1}; x_n', 1) \equiv (x_1, \ldots, x_n; *, t),$$

$$(x_1, \cdots, x_n; y, t) \equiv (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n; y, t) \text{ if } x_i = *,$$

$$(x_1, \ldots, x_n; y, t) \equiv (x_1, \ldots, x_{i-1}, x_i{}^\vee x_{i+1}, x_{i+2}, \ldots, x_n; y, t)$$

if $x_i, x_{i+1} \in \omega({}^{m-1}K)$ and

$$(x_1, \ldots, x_n; y, t) \equiv (x_1, \ldots, x_{n-1}; x_n{}^\vee l(y, t))$$

if $x_n, y \in \omega({}^{m-1}K)$. This identifications offer us a cellular decomposition of $\omega({}^mK, {}^mL)$ and a fortiori that of $\omega(K, L)$. Cells of $\omega(K, L)$ are represented by finite sequences $(\sigma_1, \ldots, \sigma_n; \tau)$ of cells $\sigma_i \in K$ and $\tau \in L$, and they have the dimensions; $\sum \dim \sigma_i + \dim \tau - n$.

Remark: Similar arguments may be treated for a standard space ${}^mX = E(X_0; f_1, \hat{X}_1; \ldots; f_m, \hat{X}_m)$, in which $\hat{X}_i$ are singular corns with bases X_i and f_i are maps of X_i into standard loop spaces $\omega({}^{i-1}X)$ of ${}^{i-1}X$. The conclusion of the following theorem II holds under a suitable smoothness condition.

§ 3. **Fundamental Theorem.** The fundamental result of this notes is stated as follows:

Theorem II. *If a pair* (K, L) *of standard complexes is* 2*-connected and if* K *is simply connected, then the injection homomorphisms of homotopy groups* $i_*: \pi_p(\omega(K)) \to \pi_p(\Omega(K))$ *and* $i_*: \pi_p(\omega(K, L)) \to \pi_p(\Omega(K, L))$ *are all isomorphisms.*

Corollary. *We have isomorphisms* $\pi_p(\omega(K)) \approx \pi_{p+1}(K)$ *and* $\pi_p(\omega(K, L)) \approx \pi_{p+1}(K, L)$.

Lemma. *For every simply connected space* X, *there exist a standard complex* K *and a map* f *of* K *into* X *inducing the isomorphisms of the homotopy groups of* K *onto those of* X.

Let $\pi_p(X; X_1, \ldots, X_{n-1})$ be the n-ad homotopy group of Blakers and Massey[7], and set $Y = X_1 \cap \ldots \cap X_{n-1}$ and $Y_i = X_1 \cap \ldots \cap X_{i-1} \cap X_{i+1} \cap \ldots \cap X_{n-1}$. If $X = Y_1 \cup \ldots \cup Y_{n-1}$, in the following exact sequence $\pi_{p+1}(X; X_1, \ldots, X_{n-1}) \xrightarrow{\partial} \pi_p(X_1; X_1 \cap X_2, \ldots, X_1 \cap X_{n-1}) \xrightarrow{i_*} \pi_p(X; X_2, \ldots, X_{n-1}) \to \pi_p(X; X, \ldots, X_{n-1}) \xrightarrow{\partial} \cdots$, the injection homomorphism i_* is *an excision* of $(n-1)$ -ad homotopy group, because $X_1 \cap X_i = X_i - (Y_1 - Y)$ $(i = 2, \ldots, n-1)$. Then the main theorem of Blakers and Massey[7] in triad homotopy group is generalized to

Proposition 1. *Let* $(X; X_1, \ldots, X_{n-1})$ *be an* n*-ad such that* $X = Y_1 \cup \ldots \cup Y_{n-1}$ *If* X *is simply connected and* (Y_i, Y) *are* r_i*-connected* $(r_i \geqq 2)$ *and if in every subpair of* $(X; X_1, \ldots; X_{n-1})$ *the excision axiom of singular homology theory holds, then we have*

$$\pi_p(X; X_1, \ldots, X_{n-1}) = 0 \qquad \textit{for } n \leqq p \leqq \sum_{i=1}^{n-1} r_i.$$

Let $K^* = K \cup \varepsilon^n$ be a complex obtained from a complex K by attaching the singular n-cells. If $(K, \dot{\varepsilon}^n)$ is m-connected and $\dot{\varepsilon}^n$ is r-connected then homomorphisms

$$P: \ \pi_n(\varepsilon^n, \dot{\varepsilon}^n) \times \pi_{p-n+1}(X, \dot{\varepsilon}^n) \to \pi_p(X^*; \varepsilon^n, X)$$

induced by the generalized Whitehead product[8] are isomorphisms for $p < m+n+r$ and homomorphisms onto for $p \leqq m+n+r$.

§4. **Homotopy Groups of Spheres.** Since the $(n+1)$-sphere S^{n+1} is a suspended space of the n-sphere S^n, the standard loop space $\omega(S^{n+1})$ may be defined, and it is constituted by kn-cells e^{kn} $(k=0, 1, 2, \dots)$ such that $e^0 \vee e^n$ is an n-sphere S^n and e^{2n} is attached to S^n by a map $[i_n, i_n]: S^{2n-1} \to S^n$ which represents Whitehead product of the identical map of S^n. The attachment of e^{3n} is represented by means of the generalized Whitehead product[8] $(2+(-1)^n)[i_n, i_{2n}]_r$ and a nullhomotopy of $\partial((2+(-1)^n)[i_n, i_{2n}]_r) = (2+(-1)^n)[i_n, [i_n, i_n]]$, where i_{2n} represents a generator of $\pi_{2n}(S^n \cup e^{2n}, S^n)$.

Proposition 2. *There are homomorphisms χ of $\pi_p(S^{2n-1}) + \pi_p(S^{3n-2} \underset{3}{\cup} e^{3n-1})$ for even n ($\pi_p(S^{2n-1})$ for odd n) into $\pi_{p+2}(S^{n+1}; E_+^{n+1}, E_-^{n+1})$ such that χ are isomorphisms for $p<4n-3$ and homomorphisms onto for $p \leqq 4n-3$, where $S^{3n-2} \underset{3}{\cup} e^{3n-1}$ is a cell-complex obtained from S^{3n-2} attaching a cell e^{3n-1} by a mapping of degree 3.*

By normalizing the complex $\omega(S^{n+1})$ to a standard form, we denote a standard loop complex $\omega(\omega(S^{n+1}), S^n)$ by Q_{n+1}. Then the homology groups of Q_{n+1} are applied to the calculation of the homotopy groups $\pi_p(Q_{n+1}) \approx \pi_{p+2}(S^{n+1}; E_+^{n+1}, E_-^{n+1})$. In case $n=3$, we have the following results;

P	3	4	5	6	7	8	9	10
$H_p(Q_3)$	Z	Z_3	0	Z_2	Z_3	Z_{15}	Z_2	0
$\pi_p(Q_3)$	Z	Z_6	Z_2	Z_{24}	Z_6	Z_{30}	Z_6	

It follows from an exact sequence

$$\cdots \to \pi_p(Q_{n+1}) \to \pi_p(S^n) \overset{E}{\to} \pi_{p+1}(S^{n+1}) \to \pi_{p-1}(Q_{n+1}) \to \cdots$$

that

Proposition 3. *We have $\pi_9(S^3) = Z_3$, $\pi_{10}(S^3) = Z_{15}$ and $\pi_{11}(S^3) = Z_2$.*

Corollary[9] i) $\pi_{10}(S^4) = Z_{24} + Z_3$, *and* $\pi_{n+6}(S^n) = Z$ *for*$_2$ $n \geqq 5$.

ii) $\pi_{11}(S^4) = Z_{15}$, $\pi_{12}(S^5) = \pi_{13}(S^6)/Z_2 = Z_{30}$,

$\pi_{14}(S^7)/G_4$[10] $= Z_{30}$, $\pi_{15}(S^8) = Z + \pi_{14}(S^7)$ *and* $\pi_{n+7}(S^n)/G_4 = Z_{60}$ *for* $n \geqq 9$.

iii) $\pi_{12}(S^4) = \pi_{13}(S^5) = Z_2$, $\pi_{14}(S^6)/Z_2 = Z_2$,

$\pi_{15}(S)/Z_2 = Z_2$, $\pi_{16}(S^8) = Z_2 + \pi_{15}(S^7)$, $\pi_{17}(S^9)/Z_2 = Z_2$ *and* $\pi_{n+8}(S^n) = Z_2$ *for* $n \geqq 10$.

References

1) $I=[0, 1]$ indicates unit interval.
2) "Homologie singulière des espace fibrés", Ann. Math., **54** (1951).
3) "On the equivalence of two singular homology theories", Ann. Math., **51** (1950).
4) $\dot{I} = (0) \cup (1)$.

5) The theorem holds for the injection homomorphisms of the singular homology groups.

6) "Borsuk's cohomotopy groups", Ann. Math., **50** (1949).

7) A. L. Blakers and W. S. Massey: "The homotopy groups of a triad I", Ann. Math., **53** (1952).

8) H. Toda: "Generalized Whitehead products and homotopy groups of spheres", Jour. Inst. Poly. Osaka City Univ., **3** (1952).

9) For $p \leqq n+5$, the groups $\pi_p(S^n)$ are obtained by J-P. Serre ("Sur la suspension de Freudenthal", C. R., **234** (1952)) and the author.

10) $G_4/Z_2 = Z_2$