113. On the Transformations Preserving the Canonical Form of the Equations of Motion

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Introduction. In this paper, we shall prove that any transformation preserving the canonical form of the equations of motion can be composed of a canonical transformation and a transformation of the form $Q_i=\rho q_i$, $P_i=p_i$ $i=1,\dots,n$ where $\rho\neq 0$ is a constant. (For the precise formulation, see section 3, 4.)

For the sake of completeness, we shall prove first some lemmas on matrices which will be used later.

1. We shall call a real regular matrix A of degree $2n$, *a real quasi-symplectic matrix* (we abbreviate it as r.q.s.m.) *with a multiplier* ρ, if

$$\rho\sum_{i=1}^{n}(x_iy_{i+n}-x_{i+n}y_i)=\sum_{i=1}^{n}(x_i'y_{i+n}'-x_{i+n}'y_i') \tag{1}$$

for two arbitrary vectors $(x_1,\dots,x_{2n})$, $(y_1,\dots,y_{2n})$, where ρ is a real number and

$$\begin{pmatrix} x_1' \\ \vdots \\ x_{2n}' \end{pmatrix}=A\begin{pmatrix} x_1 \\ \vdots \\ x_{2n} \end{pmatrix}\qquad \begin{pmatrix} y_1' \\ \vdots \\ y_{2n}' \end{pmatrix}=A\begin{pmatrix} y_1 \\ \vdots \\ y_{2n} \end{pmatrix}.$$

A r.q.s.m. with the multiplier 1 is called *a real symplectic matrix* (we abbreviate it as r.s.m.). A real regular matrix A of degree $2n$ is a r.q.s.m. with a multiplier ρ if and only if

$$\rho J=A^*JA \tag{2}$$

where A^* is the transposed of A and

$$J=\begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}\ (E_n \text{ is the unit matrix of degree } n).$$

From (2), a multiplier of a r.q.s.m. is a non-vanishing real number.

A real matrix B of degree $2n$ is called *an infinitesimal real symplectic matrix* (we abbreviate it as i.r.s.m.), if

$$JB+B^*J=0. \tag{3}$$

If we write a real matrix B of degree $2n$ in the form

$$B=\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

where B_1, B_2, B_3, B_4 are matrices of degree n, then B is an i.r.s.m. if and only if

$$B_4=-B_1^*,\qquad B_3=B_3^*,\qquad B_2=B_2^*. \tag{4}$$

2. **Lemma 1.** *Let* $A(t)$, $B(t)$ *be real matrices of degree* $2n$ *de-*

pending on a parameter t ($t_0 \leqq t \leqq t_1$) *and* $B(t)$ *be an i.r.s.m. for every t in the interval* $t_0 \leqq t \leqq t_1$. *If* $dA(t)/dt$ *exists for* $t_0 \leqq t \leqq t_1$ *and*

$$\frac{d}{dt} A(t) = B(t)\, A(t) \quad \text{for} \quad t_0 \leqq t \leqq t_1 \qquad A(t_0) = A_0$$

where A_0 *is a r.q.s.m. with a multiplier* ρ, *then* $A(t)$ *is a r.q.s.m. with the same multiplier* ρ *for any t in the interval* $t_0 \leqq t \leqq t_1$.

Proof. From

$$\frac{d}{dt} A(t) = B(t)\, A(t)$$

we have

$$\frac{d}{dt} A^*(t) = A^*(t)\, B^*(t)\,.$$

Hence

$$\frac{d}{dt} \left\{ A^*(t)\, JA(t) \right\} = \left\{ \frac{d}{dt} A^*(t) \right\} JA(t) + A^*(t) J \frac{d}{dt} A(t)$$

$$= A^*(t) \left\{ B^*(t) J + JB(t) \right\} A(t) \quad \text{for} \quad t_0 \leqq t \leqq t_1\,.$$

Then by (3), we have

$$\frac{d}{dt} \left\{ A^*(t)\, JA(t) \right\} = 0 \qquad \text{for} \quad t_0 \leqq t \leqq t_1\,.$$

On the other hand, by (2) $A^*(t_0)JA(t_0) = A_0^* JA_0 = \rho J$. Hence $A^*(t)$ $JA(t) = \rho J$ for $t_0 \leqq t \leqq t_1$ q.e.d.

Lemma 2. *Let X be a matrix of degree 2n with complex coefficients. If* $XB = BX$ *for all i.r.s.m. B of degree 2n, then X is of the form* αE_{2n}, *where* α *is a complex number and* E_{2n} *is the unit matrix of degree 2n.*

Proof. A diagonal matrix

$$\begin{pmatrix} \beta_1 & & & & & 0 \\ & \ddots & & & & \\ & & \beta_n & & & \\ & & & -\beta_1 & & \\ & & & & \ddots & \\ 0 & & & & & -\beta_n \end{pmatrix} = B'$$

where β_i $i = 1, \ldots, n$ are real numbers such that $\beta_i \neq 0$ and $\beta_i \neq \pm \beta_j$ for $i \neq j$, is an i.r.s.m. by (4) and its diagonal elements are all different between them. From $B'X = XB'$, we can easily conclude that X is a diagonal matrix.

A matrix

$$\begin{pmatrix} B_1 & E_n \\ E_n & -B_1^* \end{pmatrix} = B''$$

where B_1 is any real matrix of degree n, is an i.r.s.m. by (4). If we write

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$$

where X_1 and X_2 are diagonal matrices of degree n, the condition $B''X=XB''$ gives

$$X_2 = X_1 \qquad B_1X_1 = X_1B_1 .$$

From the second of these formulas, we can conclude easily that X_1 is a matrix of the form αE_n, since B_1 is an arbitrary real matrix of degree n. Then by the first of the above formulas, we have $X=\alpha E_{2n}$ where α is a complex number q.e.d.

Lemma 3. *Let X be a regular real matrix of degree $2n$. If XBX^{-1} is an i.r.s.m. for every i.r.s.m. B of degree $2n$, then X is a r.q.s.m.*

Proof. Let B be any i.r.s.m. of degree $2n$ and let K denote X^*JX. Then

$$KBK^{-1} = X^*J(XBX^{-1})J^{-1}(X^*)^{-1} . \tag{5}$$

By the assumption, XBX^{-1} is an i.r.s.m. Hence by (3)

$$J(XBX^{-1}) = -(XBX^{-1})^*J = -(X^*)^{-1}B^*X^*J .$$

Putting this in (5), we have

$$KBK^{-1} = -B^* .$$

On the other hand by (3)

$$JBJ^{-1} = -B^* .$$

Hence if we put $L=J^{-1}K=J^{-1}X^*JX$, we have

$$LB = BL \qquad \text{for any i.r.s.m. } B \text{ of degree } 2n .$$

Therefore by Lemma 2, L is of the form αE_{2n} where α is a real number as L is a real matrix. Then $X^*JX=\alpha J$ q.e.d.

3. We shall call a connected open set in R^n *a domain in R^n*.

In the following, we denote by G a domain in $R^{2n+1}(q_1,\dots,q_n, p_1,\dots,p_n,t)$ and by G_t, the set of the points $(q_1,\dots,q_n,p_1,\dots,p_n)$ of R^{2n} such that $(q_1,\dots,q_n,p_1,\dots,p_n,t)\in G$. G_t is open in R^{2n} for any t.

Let M denote a one to one mapping

$$(q_1,\dots,q_n,p_1,\dots,p_n,t) \to (Q_1,\dots,Q_n,P_1,\dots,P_n,t) \tag{6}$$

of G onto some domain in R^{2n+1} such that $Q_i(q_j,p_j,t)$, $P_i(q_j,p_j,t)$ are of class C^2 and the Jacobian $\partial(Q_i,P_j)/\partial(q_k,p_m)\neq 0$ on G. For such M we denote by M_t the one to one mapping

$$(q_i,p_i) \to \left\{Q_i(q_j,p_j,t), P_i(q_j,p_j,t)\right\}$$

depending on t of G_t onto some open set in R^{2n} (if $G_t\neq 0$).

We shall call M *a pseudo-canonical transformation containing the time* (we abbreviate it as p.c.t.t.) *with a multiplier* ρ, if M_t satisfies the condition

$$\rho\sum_{i=1}^{n}[dp_i\,dq_i] = \sum_{i=1}^{n}[dP_i\,dQ_i] \qquad \text{on } G_t \tag{7}$$

for every t such that $G_t\neq 0$ where ρ $(\neq 0)$ is a constant independent of q_i, p_i, t. (Here [ ] means Cartan's exterior product.) We shall call a p.c.t.t. with the multiplier 1, *a canonical transformation con-*

taining the time (we abbreviate it as c.t.t.).

We denote by $M(\rho)$ the special p.c.t.t. with a multiplier ρ

$$(q_1, \dots, q_n, p_1, \dots, p_n, t) \to (\rho q_1, \dots, \rho q_n, p_1, \dots, p_n, t).$$

Then we can easily prove the following :

Lemma 4. *Any p.c.t.t. M with a multiplier ρ can be represented as $M(\rho)M'$ where M' is a c.t.t.*

We call a system of differential equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad i = 1, \dots, n \tag{8}$$

a canonical system with a Hamiltonian $H(q_i, p_i, t)$, when $H(q_i, p_i, t)$ is defined and of class C^1 and $\partial H/\partial q_i$, $\partial H/\partial p_i$ $i=1,\dots,n$ are of class C^1 on a domain in R^{2n+1}.

Let M be a mapping of the domain G as defined in (6) and the Hamiltonian $H(q_i, p_i, t)$ of (8) be defined in a neighbourhood of a point $(q_i^0, p_i^0, t^0) \in G$. If M transforms all the integral curves of (8) in a neighbourhood of (q_i^0, p_i^0, t^0) into integral curves of another canonical system

$$\frac{dQ_i}{dt} = \frac{\partial K}{\partial P_i} \qquad \frac{\partial P_i}{dt} = -\frac{\partial K}{\partial Q_i} \qquad i = 1, \dots, n \tag{9}$$

with a Hamiltonian $K(Q_i, P_i, t)$ defined in a neighbourhood of $\{Q_i(q_j^0, p_j^0, t^0),\ P_i(q_j^0, p_j^0, t^0), t^0\}$, then we say that M *preserves the canonical form of* (8) *and transforms* (8) *into* (9), *in a neighbourhood of* (q_i^0, p_i^0, t^0). If M preserves the canonical form of every canonical system with a Hamiltonian defined on a domain $G' \subset G$, in a neighbourhood of every point belonging to G', then we say that M *preserves the canonical form* (*in* G).

It is well-known that a c.t.t. and $M(\rho)$ both preserve the canonical form[1]. Hence by Lemma 4, a p.c.t.t. preserves the canonical form. We shall prove the converse of this proposition in the following.

4. Let (q_i^0, p_i^0, t^0) be any point in the domain G and the Hamiltonian $H(q_i, p_i, t)$ of (8) be defined in a neighbourhood of (q_i^0, p_i^0, t^0). If (u_i, v_i) belongs to a neighbourhood in R^{2n} of(q_i^0, p_i^0), then we have a unique solution of (8), $q_i = \varphi_i(t, u_j, v_j)$ $p_i = \psi_i(t, u_j, v_j)$ $i=1,\dots,n$ defined for t in a neighbourhood of t^0 such that $u_i = \varphi_i(t^0, u_j, v_j)$ $v_i = \psi_i(t^0, u_j, v_j)$ $i=1, \dots, n$. We call such φ_i, ψ_i the characteristic functions of (8) at (q_i^0, p_i^0, t^0).

We denote by $S(t, u_i, v_i)$ the functional matrix of the mapping T_t: $(u_i, v_i) \to \{\varphi_i(t, u_j, v_j),\ \psi_i(t, u_j, v_j)\}$

$$\left(\begin{array}{c|c} \dfrac{\partial \varphi_i}{\partial u_j} & \dfrac{\partial \varphi_i}{\partial v_j} \\ \hline \dfrac{\partial \psi_i}{\partial u_j} & \dfrac{\partial \psi_i}{\partial v_j} \end{array}\right).$$

By the assumption that $\partial H/\partial p_i$, $\partial H/\partial q_i$ are of class C^1, we can easily prove the following[2]:

Lemma 5.

$$\left(\frac{\partial S}{\partial t}\right)_0 = \left(\begin{array}{c|c} \left(\frac{\partial^2 H}{\partial p_i \partial q_j}\right)_0 & \left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right)_0 \\ \hline -\left(\frac{\partial^2 H}{\partial q_i \partial q_j}\right)_0 & -\left(\frac{\partial^2 H}{\partial q_i \partial p_j}\right)_0 \end{array}\right) \tag{10}$$

where $(\quad)_0$ *means the value of a function for* $t=t^0$, $u_i=q_i^0$, $v_i=p_i^0$ *or for* $t=t^0$, $q_i=q_i^0$, $p_i=p_i^0$ *according to its arguments.*

Let M be a mapping of G as defined in (6). Now we assume that M preserves the canonical form. Then, in a neighbourhood of (q_i^0, p_i^0, t^0), M transforms (8) into another canonical system (9) with a Hamiltonian $K(Q_i, P_i, t)$ defined in a neighbourhood of $\{Q_i(q_j^0, p_j^0, t^0), P_i(q_j^0, p_j^0, t^0), t^0\}$. We put $Q_i^0=Q_i(q_j^0, p_j^0, t^0)$ $P_i^0=P_i(q_j^0, p_j^0, t^0)$.

If (U_i, V_i) belongs to a neighbourhood in R^{2n} of (Q_i^0, P_i^0) and t belongs to a neighbourhood of t^0, then we can define the characteristic functions of (9) at (Q_i^0, P_i^0, t^0)

$$Q_i = \Phi_i(t, U_j, V_j) \qquad P_i = \Psi_i(t, U_j, V_j) \quad i = 1, \dots, n$$

as they are defined for (8) before.

We denote by $\mathfrak{S}(t, U_i, V_i)$ the functional matrix of the mapping $\mathfrak{T}_t$: $(U_i, V_i) \to \left\{\Phi_i(t, U_j, V_j),\ \Psi_i(t, U_j, V_j)\right\}$. Then by Lemma 5

$$\left(\frac{\partial \mathfrak{S}}{\partial t}\right)_0 = \left(\begin{array}{c|c} \left(\frac{\partial^2 K}{\partial P_i \partial Q_j}\right)_0 & \left(\frac{\partial^2 K}{\partial P_i \partial P_j}\right)_0 \\ \hline -\left(\frac{\partial^2 K}{\partial Q_i \partial Q_j}\right)_0 & -\left(\frac{\partial^2 K}{\partial Q_i \partial P_j}\right)_0 \end{array}\right) \tag{11}$$

where $(\quad)_0$ denotes the value of a function for $t=t^0$, $U_i=Q_i^0$, $V_i=P_i^0$ or for $t=t^0$, $Q_i=Q_i^0$, $P_i=P_i^0$ according to its arguments.

From the assumption that M transforms (8) into (9) in a neighbourhood of (q_i^0, p_i^0, t^0), it follows easily that

$$M_t T_t M_{t^0}^{-1}(U_i, V_i) = \mathfrak{T}_t(U_i, V_i) \tag{12}$$

for any (U_i, V_i, t) in a neighbourhood of (Q_i^0, P_i^0, t^0).

Let us denote by $N(t, q_i, p_i)$ the functional matrix of the mapping M_t: $(q_i, p_i) \to \left\{Q_i(g_j, p_j, t), P_i(q_j, p_j, t)\right\}$. Then by (12)

$$N(t, q_i, p_i) S(t, q_i^0, p_i^0) \left\{N(t^0, q_i^0, p_i^0)\right\}^{-1} = \mathfrak{S}(t, Q_i^0, P_i^0) \tag{13}$$

for any t in a neighbourhood of t^0, where

$$q_i = \varphi_i(t, q_j^0, p_j^0) \qquad p_i = \psi_i(t, q_j^0, p_j^0)\,.$$

If we differentiate both sides of (13) with respect to t and put $t=t^0$, then we have

$$\left(\frac{\partial N}{\partial t}\right)_0 (N)_0^{-1} + \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i}\right)_0 \left(\frac{\partial N}{\partial q_i}\right)_0 (N)_0^{-1} - \sum_{i=1}^n \left(\frac{\partial H}{\partial q_i}\right)_0 \left(\frac{\partial N}{\partial p_i}\right)_0 (N)_0^{-1}$$
$$+ (N)_0 \left(\frac{\partial S}{\partial t}\right)_0 (N)_0^{-1} = \left(\frac{\partial \mathfrak{S}}{\partial t}\right)_0, \tag{14}$$

considering that $q_i^0=\varphi_i(t^0, q_j^0, p_j^0)$ $p_i^0=\psi_i(t^0, q_j^0, p_j^0)$ and $(\partial\varphi_i/\partial t)_0=(\partial H/\partial p_i)_0$, $(\partial\psi_i/\partial t)_0=-(\partial H/\partial q_i)_0$, $(S)_0=E_{2n}$.

In (14), the right side $(\partial\mathfrak{S}/\partial t)_0$ is always an i.r.s.m. by (11), (4).

If we take $-\sum_{i=1}^{n} a_i q_i+\sum_{i=1}^{n} b_i p_i$ as H, then $(\partial H/\partial q_i)_0=-a_i$ $(\partial H/\partial p_i)_0=b_i$ and $(\partial S/\partial t)_0=0$ by (10). Hence by (14)

$$\left(\frac{\partial N}{\partial t}\right)_0 (N)_0^{-1}+\sum_{i=1}^{n} b_i\left(\frac{\partial N}{\partial q_i}\right)_0 (N)_0^{-1}+\sum_{i=1}^{n} a_i\left(\frac{\partial N}{\partial p_i}\right)_0 (N)_0^{-1}=\left(\frac{\partial\mathfrak{S}}{\partial t}\right)_0$$

where a_i, b_i are arbitrary real numbers. Hence $(\partial N/\partial t)_0(N)_0^{-1}$, $(\partial N/\partial q_i)_0(N)_0^{-1}$, $(\partial N/\partial p_i)_0(N)_0^{-1}$ are i.r.s.m. From this by (14), $(N)_0(\partial S/\partial t)_0(N)_0^{-1}$ is always an i.r.s.m. and by (10) if we take a suitable quadratic form in p_i, q_i as H, we can turn $(\partial S/\partial t)_0$ into an arbitrary i.r.s.m. Hence by Lemma 3, $(N)_0$ is a r.q.s.m.

Thus we have proved that $N(q_i,p_i,t)$ is a r.q.s.m. and $(\partial N/\partial t)N^{-1}$, $(\partial N/\partial p_i)N^{-1}$, $(\partial N/\partial q_i)N^{-1}$ are i.r.s.m. for any point $(q_i, p_i, t)\in G$. From this we can prove easily that $(dN/ds)N^{-1}$ is an i.r.s.m. along any curve $q_i=q_i(s)$, $p_i=p_i(s)$, $t=t(s)$ $s_0\leqq s\leqq s_1$ in G with continuous $q_i'(s)$, $p_i'(s)$, $t'(s)$. On the other hand N is a r.q.s.m. for any $(q_i, p_i, t)\in G$. Hence by Lemma 1, N is a r.q.s.m. with the same multiplier along any such curve.

Since G is a domain, we can join any two of its points by a polygonal line. Therefore $N(q_i, p_i, t)$ is a r.q.s.m. with the same multiplier ρ for any $(q_i, p_i, t)\in G$. This means by (1), (7) that M is a p.c.t.t. Thus we have proved the following:

Theorem. *Let M be a one to one mapping $(q_i, p_i, t)\rightarrow(Q_i, P_i, t)$ of a domain G in R^{2n+1} onto some domain in R^{2n+1} with $Q_i(q_j, p_j, t)$, $P_i(q_j, p_j, t)$ of class C^2 and with the Jacobian $\partial(Q_i, P_j)/\partial(q_k, p_m)\neq 0$ on G. M preserves the canonical form in G if and only if M is a pseudo-canonical transformation containing the time.*

By this theorem and Lemma 4, we have determined the form of the transformations preserving the canonical form of the equations of motion.

References

1) Cf. Handbuch der Physik, **5**, 97–100 (1927) (Julius Springer, Berlin).
2) Cf. E. Kamke, Differentialgleichungen Reller Funktionen (1930), § 18, Satz 1.