97. An Observation on the Brown-McCoy Radical

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We wish to characterize in this note the Brown-McCoy radical $G(A)$ of an associative ring A, as a radical $(1,1,1,1)(A)$, $(1,1,1,0)(A)$, $(1,1,0,1)(A)$ and $(1,2,1,1)(A)$, respectively, where $(k,l,m,n)(A)$ is a well-defined special F-radical of the ring A in the sense of Brown-McCoy [3] for arbitrary nonnegative integers k, l, m and n. The concept of a (k,l,m,n)-radicalring A can be illustrated by the following elementary remarks. If the elements of A form on the operation $a\circ b=a+b-ab\,(a,b\in A)$ a Neumann-regular semigroup (for instance in the case of a Jacobson-radicalring A, when $(A,0)$ is a group), then A is a $(k,0,1,1)$-radicalring and a $(0,l,1,1)$-radicalring at the same time for any integers $k,l\geqq 0$. Furthermore any (k,l,m,n)-semisimple ring A with minimum condition on *twosided* principal ideals is, as an (A,A)-doublemodule, completely reducible in a weak meaning, which generalizes the classical Wedderburn-Artin structure theorem also. (For the details of radicals, see [1], [2], [3].)

In this note the knowing of the results of Brown-McCoy [3] will be assumed for the reader. We denote the sum of all *twosided* principal ideals $(a^{(m)}\circ x\circ a^{(n)}-k\cdot a^{(l)})$ by $(k,l,m,n)(a)$, where a is a fixed element, X a varying element of A, $a\circ b=a+b-ab$, $a^{(0)}=0$, $a^{(1)}=a$, $a^{(k+1)}=a^{(k)}\circ a$ and k,l,m,n are nonnegative integers. An element $a\in A$ is called *(k,l,m,n)-regular*, if $a\in(k,l,m,n)(a)$. We call an element $a\in A$ *strictly (k,l,m,n)-regular*, if any element b of the *twosided* principal ideal (a) generated by a is *(k,l,m,n)-regular*. The set $(k,l,m,n)(A)$ of all strictly (k,l,m,n)-regular-elements of A is called the (k,l,m,n)-radical of A. *This is evidently a special F-radical of A* [3]. The rings with (k,l,m,n)-radical (0) are called (k,l,m,n)-semisimple. We call a subdirectly irreducible (k,l,m,n)-semisimple ring A shortly: (k,l,m,n)-primitive. An element $a\neq 0$ with the condition $(k,l,m,n)(a)=0$ is called here a (k,l,m,n)-distinguished element of A. By [3] the (k,l,m,n)-radical of A is the intersection of such ideals $\mathfrak{T}_\gamma\,(\gamma\in\Gamma)$ of A, that the factorrings $A/\mathfrak{T}_\gamma$ are (k,l,m,n)-primitive. $A/(k,l,m,n)(A)$ is (k,l,m,n)-semisimple, and a subdirect sum of (k,l,m,n)-primitive rings. By [3] a subdirectly irreducible ring A is (k,l,m,n)-primitive if and only if the minimal ideal $\mathfrak{D}\neq 0$ of A contains a (k,l,m,n)-distinguished element $d\neq 0$ playing the role of unity element in the case of radical

$(1,1,1,1)(A)=G(A)$ of A.

Then holds the following

Theorem. An arbitrary (k,l,m,n)-primitive ring P has no proper twosided ideals, and we have $(1-d^{(m)})P(1-d^{(n)})=0$, $d=kd\cdot d^{(l)}$, $kd^{(l)}=d^{(m+n)}$ for a (k,l,m,n)-distinguished element $d(\neq 0)$ of P. Furthermore $G(A)=(1,1,1,1)(A)=(1,1,1,0)(A)=(1,1,0,1)(A)=(1,2,1,1)(A)$ are valid for the Brown-McCoy radical $G(A)$ of an arbitrary (*associative*) ring A.

Proof. If P is (k,l,m,n)-primitive, then there exists [3] a (k,l,m,n)-distinguished element $d\neq 0$ in the minimal ideal $\mathfrak{D}\neq 0$ of P. We have from $(k,l,m,n)(d)=0$ evidently $d^{(m)}\circ x\circ d^{(n)}=k\cdot d^{(l)}$ for any $x\in P$. In the special case $X=0$ follows $d^{(m+n)}=kd^{(l)}$ and thus in the case of arbitrary $x\in P$ is $X=d^{(m)}\cdot x+xd^{(n)}-d^{(m)}xd^{(n)}\in\mathfrak{D}$ valid. Therefore one has $P=\mathfrak{D}$ for the (k,l,m,n)-primitive rings P, and thus P cannot have proper twosided ideals. Obviously follows also $(1-d^{(m)})P(1-d^{(n)})=0$, $d=d\cdot d^{(m+n)}$ and $d=kd\cdot d^{(l)}$ respectively. Let A be now an arbitrary *associative* ring. Then $(1,1,1,1)(A)=G(A)$ will be proved by showing, that any $(1,1,1,1)$-primitive ring P is a simple ring with unity element, and a similar fact holds for other special k, l, m, n mentioned in the above theorem. In the four cases k,l,m,n mentioned above, $k=1$, hence $d=d\cdot d^{(l)}$ and $d^{(l)}=d^{(m+n)}$. If $l=m=n=1$, then one has $d^2=d$ for the (k,l,m,n)-distinguished element $d\neq 0$ of the (k,l,m,n)-primitive ring P. By $(1-d)P(1-d)=0$ follows $C=(1-d)P+P(1-d)P=0$, since P is by $d^2=d\neq 0$ semi-simple in the sense of Jacobson, and the ideal C is nilpotent. Thus $(1-d)P=0$, $P=dP\,(d^2=d)$ and similarly $P=Pd$ too. Therefore one has $(1,1,1,1)(A)=G(A)$. If $k=l=m=1$ and $n=0$, immediately follows

$$(1,1,1,0)(a)=\sum_{x\in A}(a\circ x\circ a^{(0)}-a)=\sum_{x\in A}(X-ax)=(1-a)A+A(1-a)A,$$

and thus $(1,1,1,0)(A)=G(A)$ by the definition of the Brown-McCoy radical $G(A)$ of A [3]. The case $k=l=n=1$ and $m=0$ is totally similar to the previous case. If $k=m=n=1$ and $l=2$, then one has $d=d\cdot d^{(2)}$ and thus $d-2d^2+d^3=0$. Then by $d=2d^2-d^3\neq 0$ is surely $P^2\neq 0$, i.e. P is semisimple in the sense of Jacobson by the want of proper ideals. By $(1-d)P(1-d)=0$ and $P^2\neq 0$ follows $C=(1-d)P+P(1-d)P=0$, since C is a nilpotent twosided ideal of P. This means $(1-d)P=0$ and $P=dP$. From $(d-d^2)P=(1-d)dP=0$ follows by $P^2\neq 0$ evidently $d^2=d$, for a Jacobson-semisimple ring we have no annullator $\neq 0$. Therefore d is a left unity element of $P(=dP)$, and similarly one has $P=Pd$ also, which proves the theorem.

Remarks. 1) Any (k,l,m,n)-semisimple ring with minimum condition on *twosided* principal ideals is the discrete direct sum of (k,l,m,n)-primitive rings (see for these rings the above theorem), and conversely.

2) If the elements of A form with the operation $a\circ b=a+b-ab$ a Neumann-regular semigroup, then A is a $(k, 0, 1, 1)$-radicalring and a $(0, l, 1, 1)$-radicalring too.

3) It can be proved $A=(0, 0, 0, 0)(A)=(k, 0, 0, 1)(A)=(0, l, 0, 1)(A)=(k, 0, 1, 0)(A)=(0, l, 1, 0)(A)=(2, 1, 1, 0)(A)=(2, 1, 0, 1)(A)=(2, 1, 1, 1)(A)$.
For instance, if P is a $(2, 1, 1, 1)$-primitive ring, then holds $d^{(2)}=2d^{(1)}$ and $(1-d)P(1-d)=0$, consequently $2d-d^2=2d$, $d^2=0$ and $0\neq d=d-2d^2+d^3=(1-d)d(1-d)\in(1-d)P(1-d)=0$, which is a contradiction. Therefore $P=0$ and $(2, 1, 1, 1)(A)=A$.

4) Any $(k, 0, 1, 1)$-primitive ring P and any $(0, l, 1, 1)$-primitive ring P are simple rings with unity element and with the condition $2P=P\neq 0$.

5) Any $(3, 1, 1, 1)$-primitive ring, any $(3, 1, 1, 0)$-primitive ring and any $(3, 1, 0, 1)$-primitive ring P are simple rings with unity element and with the condition $2P=0$. Therefore for example a $(3, 1, 1, 1)$-primitive ring $P\neq 0$ cannot be for instance a $(0, l, 1, 1)$-primitive ring.

6) We have seen $(1, 2, 1, 1)(A)=G(A)$. Then holds $(1, 2, 1, 1)(a)=((1-a)A(1-a))=(1-a)A(1-a)+A(1-a)A(1-a)+(1-a)A(1-a)A+A(1-a)A(1-a)A\supseteq W(a)=A(1-a)A(1-a)A$. The following W-regularity: $b\in W(b)$ determines a special F-radical $W(A)$ of A. If P is a W-primitive ring *i.e. a W-semisimple and subdirectly irreducible ring*, and if $P^3\neq 0$, then P is a simple ring with unity element. If P is a W-primitive ring and if $P^2=0$, then the additive group P^+ is isomorphic to a group $C(p^k)$, where $1\leqq k\leqq\infty$. If finally $P^2\neq 0$ but $P^3=0$, and P is a W-primitive ring, then we have $P\mathfrak{D}=\mathfrak{D}P=0$ for the minimal ideal $\mathfrak{D}$ of P and $(P^2)^+\cong C(p^k)$ holds $(1\leqq k\leqq\infty)$. For example $A=\{a_1, a_2, \cdots; b_1, b_2, \cdots\}$ with $a_i^2-b_i=pa_1=b_i-pb_{i+1}=a_ia_j=a_i^3=0$ is a W-primitive ring with $A^3=0$ and $A^2\neq 0$, $(A^2)^+\cong C(p^\infty)$ $(i\neq j)$.

7) Let A be an associative ring, M a right A-module and $\mathfrak{M}$ an arbitrary cardinal number. An A-submodule K of M is called $\mathfrak{M}$-homoperfect, if the following conditions are satisfied:

I) $MA+K=M$;

II) M/K is a completely reducible A-module of dimension $<\mathfrak{M}$;

III) M/K has no proper A-submodule, which is invariant for all A-endomorphism of M/K;

IV) if φ is an A-homomorphism of M/L onto M/K for an A-submodule L with the conditions I), II) and III), then φ is an isomorphism.

Let $\mathfrak{R}_m(M)$ be now itself M, if M has no proper $\mathfrak{M}$-homoperfect submodules. If there exist in M proper $\mathfrak{M}$-homoperfect submodules $K_\gamma(\gamma\in\Gamma)$, then we define $\mathfrak{R}_m(M)=\bigcap_\gamma K_\gamma$. In the case of $1\in A$, a unitary

A-module M and $\mathfrak{M}=2$; $\mathfrak{R}_m(M)$ is the Bourbaki-radical of M [2], and in the case $\mathfrak{M}=2$ and arbitrary A we obtain the Kertész-radical of M [5]. We have proved solving in [6] a problem of Dr. A. Kertész [5] that the Jacobson-radical $\mathfrak{F}(A)$ of A must not coincide with the radical $\mathfrak{R}_2(A)$ of the right A-module A, if the power $|A|$ of A is no quadratfree finite cardinal number. We have generally only $\mathfrak{R}_2(A)\subseteq\mathfrak{F}(A)$. If in the ring A with left unity element holds the minimum condition on principal right ideals [7] and $\mathfrak{M}=\aleph_0$, then one has evidently $\mathfrak{R}_{\aleph_0}(A)\subseteq G(A)$ for the above radical $\mathfrak{R}_m(A)$ of the right A-module A and the Brown-McCoy radical $G(A)$ of A.*) Now we arise the following

Problem. What is a necessary and sufficient condition concerning A for the validity of $\mathfrak{R}_{\aleph_0}(A)=G(A)$? (*Solve a similar problem of A. Kertész on* $\mathfrak{R}_2(A)$ *and* $\mathfrak{F}(A)$ *too!*)

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*) It may be remarked that the theory of F-radicales can be formulated for A-modules too, where F is a well-defined mapping of any A-module M onto a set of submodules $F(m)$ of M ($m\in M$, $F(m)\subseteq M$) with the condition $F(m)\varphi=F(m\varphi)$ for any A-homomorphism φ of M. Then $m\in M$ is F-regular in the case $m\in F(m)$. Then the F-radical of M is the set $[m;\ m\in M,\ n\in F(n),\ n\in\{m\}]$.