91. On the Equivalence of Excessive Functions and Superharmonic Functions in the Theory of Markov Processes. I

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It is well known that the family of nonnegative superharmonic functions in the classical sense coincides with that of excessive functions associated with a Brownian motion except the fnnction of identical infinity. In this paper we shall give the exact counterpart of the above result for a class of Markov processes called regular step processes. A generalization to more general Markov processes will be discussed in the ensuing paper.

1. **Strict Markov processes.** Let S be a locally compact separable Hausdorff space and $\tilde{S}$, the space obtained by adding a death point ∞ to S as an isolated point. The topological Borel field over $\tilde{S}$ is denoted by $\tilde{\mathscr{B}}$ and a σ-finite measure over $(\tilde{S}, \tilde{\mathscr{B}})$, by μ. Define $\mathscr{B}=\bigcap_{\text{any } \mu}\{\mu\text{-completion of } \tilde{\mathscr{B}}\}$. W is the set of all the mappings w from $[0, +\infty]$ to $\tilde{S}$ which satisfies the following conditions: (1) $x_t(w)$[1] is right continuous in t, (2) $x_{+\infty}(w)=\infty$, (3) $x_t(w)=\infty$ for every $t\geqq\sigma_\infty(w)=\inf\{t\geqq 0, x_t(w)=\infty\}$ and (4) there exists $\lim_{\varepsilon\to 0} x_{t-\varepsilon}(w)$ for every $t<\sigma_\infty(w)$. For each w and t, the shifted path w_t^+ and the stopped one w_t^- are defined as follows: $x_{t'}(w_t^+)=x_{t+t'}(w)$, $x_{t'}(w_t^-)=x_{\min(t,t')}(w)$ for $t'\neq+\infty$ and $x_{+\infty}(w_t^-)=\infty$. Let $\tilde{\mathfrak{B}}$ be the Borel field of W generated by all Borel cylinder sets $\{w, x_t(w)\in A\}$ $(A\in\tilde{\mathscr{B}})$ and $\sigma(w)$, a $\tilde{\mathfrak{B}}$-measurable random time. Then both $\varphi_\sigma(w)=w_\sigma^-$ and $\psi_\sigma(w)=w_\sigma^+$ are measurable mappings from $(W, \tilde{\mathfrak{B}})$ into itself. We define $\tilde{\mathfrak{B}}_\sigma=\varphi_\sigma^{-1}(\tilde{\mathfrak{B}})$ and $\tilde{\mathfrak{B}}_{\sigma+}=\bigcap_{\varepsilon>0}\tilde{\mathfrak{B}}_{\sigma+\varepsilon}$. A function $P_x(B)$ defined on $\tilde{S}\times\tilde{\mathfrak{B}}$ is called a *strict Markov process on* S and is denoted by X if it satisfies the following conditions: $(\widetilde{X.1})$ $P_x(\cdot)$ is a probability measure on $\tilde{\mathfrak{B}}$ for any fixed x and $P_x\{x_0(w)=x\}=1$ for every x, $(\widetilde{X.2})$ $P.(B)$ is $\mathscr{B}$-measurable for every $B\in\tilde{\mathfrak{B}}$ and $(\widetilde{X.3})$ for any $B\in\tilde{\mathfrak{B}}$ and for any $\tilde{\mathfrak{B}}$-*measurable Markov time* $\sigma(w)$,[2] $P_x\{P_x(w_\sigma^+\in B\mid\tilde{\mathfrak{B}}_{\sigma+})=P_{x_\sigma}(B)\}=1$ for every x. We now define $P_\mu(B)$

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1) $x_t(w)$ expresses the value of w at $t\in[0, +\infty]$.

2) That is, $\{\sigma(w)\geqq t\}\in\tilde{\mathfrak{B}}_t$ for every t.

$=\int_{\tilde{S}} P_x(B)\mu(dx)$ and $\mathfrak{B}=\bigcap_{\text{any}\,\mu}\{P_\mu\text{-completion of }\tilde{\mathfrak{B}}\}$. Next taking a $\mathfrak{B}$-measurable random time $\sigma(w)$, put $\tilde{\mathfrak{B}}_\sigma=\varphi_\sigma^{-1}(\tilde{\mathfrak{B}})$ $(\subset\mathfrak{B})$, $\mathfrak{B}_\sigma=\bigcap_{\text{any}\,\mu}\{P_\mu$-completion of $\tilde{\mathfrak{B}}_\sigma\}$ and $\mathfrak{B}_{\sigma+}=\bigcap_{\varepsilon>0}\mathfrak{B}_{\sigma+\varepsilon}$. The conditions $(\widetilde{X.1})-(\widetilde{X.3})$ imply the following stronger conditions: $(X.1)$ $P_x(\cdot)$ is extended to a probability measure on $\mathfrak{B}$ and $P_x(x_0(w)=x)=1$ for every x, $(X.2)$ $P_\cdot(B)$ is $\mathscr{B}$-measurable for every $B\in\mathfrak{B}$ and $(X.3)$ for any $B\in\mathfrak{B}$ and for any ($\mathfrak{B}$-*measurable*) *Markov time* $\sigma(w)$,[3] $P_x\{P_x(w_\sigma^+\in B\mid\mathfrak{B}_{\sigma+})=P_{x_\sigma}(B)\}=1$ for every x.[4]

Let f be a $\mathscr{B}$-measurable function on S[5] and $f(\infty)=0$ by definition. The *Green operator* G_α $(\alpha\geqq 0)$ of X is defined by $G_\alpha f(x)=E_x\left(\int_0^\infty e^{-\alpha t}f(x_t)\,dt\right)$. For any $\beta\geqq 0$, there exists uniquely a strict Markov process $X^{(\beta)}$ (called the *β-subprocess of X*) whose Green operator $G_\alpha^{(\beta)}$ is given by $G_{\alpha+\beta}$ for every $\alpha\geqq 0$. For any $\mathfrak{B}$-measurable random time σ, put $H_\sigma f(x)=E_x(f(x_\sigma))$. A subset A of S is called *admissible* if $\sigma_A(w)=\inf\{t\geqq 0, x_t\in A\}$ is a $\mathfrak{B}$-measurable Markov time. Any open set of S is admissible. A $\mathscr{B}$-measurable function u on S is called *excessive* if $H_t u\leqq u$ for every t and $H_t u\to u$ $(t\to 0)$. Any excessive function is nonnegative.[6]

Take an open base $\mathcal{U}$ of S. A nonnegative and $\mathscr{B}$-measurable function u on S is called *$\mathcal{U}$-superharmonic* if, for any $U\in\mathcal{U}$,

(1.1) $$u(x)\geqq H_{\bar{U}^c}u(x)^{7)}\quad\text{for every } x \text{ of } U.$$

A $\mathcal{U}$-superharmonic function u is called *$\mathcal{U}$-harmonic* if it is finite-valued and if it satisfies the equality in (1.1) for every x of U such that $P_x(\sigma_{\bar{U}^c}<+\infty)=1$. When u is $\mathcal{U}$-superharmonic ($\mathcal{U}$-harmonic) for some $\mathcal{U}$, we shall call it *superharmonic* (*harmonic*). It should be noted that our definition of superharmonic or harmonic functions is confined to nonnegative functions.

2. Excessive functions. We shall here summarize some basic results on excessive functions to be used later. Most of them were proved by Hunt [3]. As for some new results, we need only simple modification of Hunt's method, so that the proof will be omitted.

2.1. *If f is a nonnegative $\mathscr{B}$-measurable function on S, the function $u=G_0 f$ is excessive. Such u is called G_0-potential.*

2.2. *If u_n is excessive and $u_n\uparrow u$, then u is excessive.*

2.3. *If both u and v are excessive, then $u\wedge v$[8] is excessive.*

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3) That is, $\{\sigma(w)\geqq t\}\in\mathfrak{B}_t$ for every t.

4) For the proof, the reader should be referred to Dynkin's book [2].

5) We shall admit f to take $\pm\infty$ as its value.

6) Put $t=+\infty$.

7) For an admissible set A, we use the notation $H_A u$ instead of $H_{\sigma_A}u$.

2.4. *The following three conditions are equivalent to each other.* (1) *u is excessive,* (2) *$\alpha G_\alpha u \leqq u$ $(\alpha \geqq 0)$ and $\lim_{\alpha \to +\infty} \alpha G_\alpha u = u$.* (3) *$H_\sigma u \leqq u$ for any Markov time σ, and $H_{\sigma_n} u(x) \to u(x)$ for any x of S and for any sequence of Markov times σ_n such that $P_x(\sigma_n \downarrow 0)=1$.*

2.5. *If u is excessive and if the Markov times σ_n, σ satisfy the condition that $\sigma_n(w) \downarrow \sigma(w)$ with P_x-probability 1, $H_{\sigma_n} u(x) \uparrow H_\sigma u(x)$.*

3. **Regular step processes.** Let X be a strict Markov process. It is shown that the *first jumping time* $\sigma_1(w) = \inf \{t \geqq 0, x_t(w) \neq x_0(w)\}$ is a Markov time. Therefore the *n-th jumping time* $\sigma_n(w) = \sigma_{n-1}(w) + \sigma_1(w^+_{\sigma_{n-1}})$ is also a Markov time for every n. Put $q(x) = [E_x(\sigma_1)]^{-1}$ and $\prod(x, A) = P_x\{x_{\sigma_1} \in A\}$, where x is a point of S and A is a $\mathscr{B}$-measurable set of S. It is well known that $P_x(\sigma_1 > t) = \exp\{-q(x)t\}$. The state point x of S is called *trap, sojourn state* or *instantaneous state* according as $q(x)=0$, $0<q(x)<+\infty$ or $q(x)=+\infty$.

3.1. *If x is a sojourn state, $\prod(x, x)=0$.*

For the brevity, write $\sigma(w)$ instead of $\sigma_{\{x\}^c}(w)$. Since $\sigma(w)=\sigma_1(w)$ with P_x-probability 1, we have $P_x(\sigma>0)=1$ by the assumption and therefore $\prod(x, x) = P_x(x_\sigma = x) = P_x(x_\sigma = x, \sigma > 0)$. On the other hand, $P_x\{\sigma(w^+_\sigma) > 0\} = 0$ from the definition of σ. Using the strict Markov property, we get

$$\begin{aligned} 0 &= P_x\{\sigma(w^+_\sigma)>0\} \geqq P_x\{x_\sigma = x,\ x_\sigma(w^+_\sigma)=x,\ \sigma(w^+_\sigma)>0\} \\ &= E_x\{P_{x_\sigma}(x_\sigma = x,\ \sigma>0);\ x_\sigma = x\} = P_x(x_\sigma = x,\ \sigma > 0) P_x(x_\sigma = x) \\ &= [\prod(x,x)]^2 \geqq 0. \end{aligned}$$

A strict Markov process is called a *regular step process* (RSP), if it satisfies the conditions that every point in S is a sojourn state and that $\lim_{n \to \infty} \sigma_n(w) \geqq \sigma_\infty(w)$ with P_x-probability 1 for any x. These conditions mean that almost all the paths of an RSP are step functions, and therefore we can easily establish the potential theory of the RSP by the same method as in the case of countable Markov processes (see Doob [1] and the author [4]). We shall now introduce several notions concerning RSP's. The system $(q, \prod)$ induced by an RSP X is called the *canonical system* of X and the operator $\mathscr{G}_D f(x) = q(x)[\prod f(x) - f(x)]$, the *Dynkin generator*. A nonnegative $\mathscr{B}$-measurable function u is called $\prod$-*superharmonic* if $\mathscr{G}_D u \leqq 0$ (or equivalently $\prod u \leqq u$), and $\prod$-*harmonic* if it is finite-valued and $\mathscr{G}_D u = 0$ (or equivalently $\prod u = u$).

3.2. *The canonical system of an RSP satisfies the following conditions:* (1) *Both $q(\cdot)$ and $\prod(\cdot, A)$ (for any fixed A) are $\mathscr{B}$-measurable functions on S,* (2) *$0<q(x)<+\infty$,* (3) *$\prod(x,x)=0$ and* (4) *$\prod(x, \cdot)$ is a measure on S whose total mass does not exceed* 1. *Conversely any $(q, \prod)$-system satisfying the conditions* (1)–(4) *is the*

8) $u_\wedge v(x) = \min(u(x), v(x))$.

canonical system of one and only one RSP.

3.3. *The Green operator G_α ($\alpha \geqq 0$) of an RSP is expressed in the following form:*

$$G_\alpha f = \sum_{n=0}^{\infty} [(\alpha+q)^{-1} q \prod]^n (\alpha+q)^{-1} f$$
$$= \sum_{n=0}^{\infty} (\alpha+q)^{-1} [q \prod (\alpha+q)^{-1}]^n f$$

3.4. *The β-subprocess $X^{(\beta)}$ of the RSP is also an RSP and its canonical system $(q^{(\beta)}, \prod^{(\beta)})$ is given by $q^{(\beta)}(x) = \beta + q(x)$ and $\prod^{(\beta)}(x, A) = [\beta + q(x)]^{-1} q(x) \prod(x, A)$. Moreover its Dynkin generator $\mathcal{G}_D^{(\beta)}$ equals $(\mathcal{G}_D - \beta)$.*

The proofs of these three propositions are quite analogous to those in the case of countable Markov processes (see [4]) and therefore will be omitted.

3.5. *Any finite-valued $\prod$-superharmonic function u is decomposed uniquely in the form of $u = G_0 f + v$, where f is a nonnegative $\mathcal{B}$-measurable function and v is a $\prod$-harmonic function. In fact, f is given by $-\mathcal{G}_D u$ and v, by $\lim_{n\to\infty} \prod^n u$.*

Noting that u is finite-valued and $\prod^n u$ decreases monotonely with n, it is clear that $\lim_{n\to\infty} \prod^n u$ exists and is $\prod$-harmonic. Further we have

$$\sum_{n=0}^{N} \prod^n q^{-1} (-\mathcal{G}_D) u = \sum_{n=0}^{N} \prod^n [u - \prod u] = u - \prod^{N+1} u.$$

Letting $N \to +\infty$, $G_0(-\mathcal{G}_D)u = u - \lim_{N\to\infty} \prod^{N+1} u$, which proves the existence of the required decomposition. Next we shall show the uniqueness. Suppose that $u = G_0 f + v$. Since this $G_0 f$ is finite-valued, $\prod^n G_0 f(x) = E_x\left(\int_{\sigma_n}^{\sigma_\infty} f(x_t)\, dt\right)$ tends to zero monotonely with n. Therefore $\prod^n u = \prod^n G_0 f + \prod^n v = \prod^n G_0 f + v \downarrow v$. From the arguments in the proof of the existence, we have $G_0 f = G_0(-\mathcal{G}_D)u$. But, in general, f is uniquely determined by $G_0 f$.[9] Consequently, $f = -\mathcal{G}_D u$.

The following two propositions result immediately from 3.5.

3.6. *Suppose that X is an RSP. Then a finite-valued $\mathcal{B}$-measurable function u on S is a G_0-potential of X if and only if it is nonnegative and $\prod^n u \downarrow 0$ ($n \to \infty$).*

3.7. *Let f be a nonnegative $\mathcal{B}$-measurable function on S. In order that the equation $-\mathcal{G}_D u = f$ should have at least one nonnegative and finite-valued solution, it is necessary and sufficient that $G_0 f$ is finite-valued. In this case, $G_0 f$ is the smallest among nonnegative solutions of the above equation.*

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9) In fact, $G_0 f(x) = E_x\left(\int_0^{\sigma_1} f(x_t)\, dt\right) + E_x\left(\int_{\sigma_1}^{\sigma_\infty} f(x_t)\, dt\right) = q(x)^{-1} f(x) + \prod G_0 f(x)$, so that $f(x) = -\mathcal{G}_D(G_0 f)$.

4. The equivalence of excessive functions and superharmonic functions for regular step processes. Our main result is now stated in.

THEOREM. *Let X be an RSP. Then u is excessive if and only if it is superharmonic, or if and only if it is $\prod$-superharmonic.*

It is implied in Proposition 2.4 that any excessive function is superharmonic. Suppose now that u is superharmonic with respect to the open base $\mathcal{U}$. For any fixed x in S, taking U_k such that $U_k \in \mathcal{U}$ and $\overline{U}_k \downarrow x$ and putting $\tau_k = \sigma_{\overline{U}_k^c}$, we get $P_x\{\tau_k = \sigma_1$ for some $k_0 \leqq$ any $k\} = 1$ (from 3.1). This shows that $\lim_{k\to\infty} f(x_{\tau_k}) = f(x_{\sigma_1})$ with P_x-probability 1 for every function f on S. Moreover it is easily verified that $u_n = u \wedge n$ is also $\mathcal{U}$-superharmonic and therefore we have

$$u_n(x) \geqq \liminf_{k\to\infty} E_x(u_n(x_{\tau_k})) \geqq E_x(\lim_{k\to\infty} u_n(x_{\tau_k})) = \prod u_n(x).$$

Letting $n \to +\infty$, $u(x) \geqq \prod u(x)$, which proves that every superharmonic function is $\prod$-superharmonic. Finally suppose that u is $\prod$-superharmonic. Clearly $u_n = u \wedge n$ is also $\prod$-superharmonic. Considering the β-subprocess $X^{(\beta)}$ of X, we have $-\mathcal{G}_D^{(\beta)} u_n = (\beta - \mathcal{G}_D) u_n \geqq \beta u_n$. Applying Proposition 3.7 to $X^{(\beta)}$ and recalling Proposition 3.4, $u_n \geqq G_0^{(\beta)}[-\mathcal{G}_D^{(\beta)} u_n] \geqq \beta G_0^{(\beta)} u_n \geqq \beta G_\beta u_n$. Since X is the RSP and u_n is bounded, we get

$$\beta G_\beta u_n(x) = E_x\left(\int_0^\infty e^{-t} u_n(x_{t/\beta})\, dt\right) \to u_n(x),$$

which shows that u_n is excessive (Proposition 2.4). Therefore u is also excessive (Proposition 2.2). Thus our theorem has been proved completely.

In conclusion we shall add one comment. Since it is shown that every RSP satisfies the *quasi-continuity from the left*,[10] any Borel subset A of S is admissible.[11] Combining Proposition 2.5 and the theorem just proved, we see that the relation $u \geqq H_A u$ holds for every Borel subset A of S[12] if u is superharmonic (or $\prod$-superharmonic).

References

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[3] G. A. Hunt: Markoff processes and potentials I, Ill., J. Math., **1**, 44-93 (1957).

[4] T. Watanabe: On the theory of Martin boundaries induced by countable Markov processes, Mem. Coll. Science, Univ. Kyoto, Ser. A, **33**, 39-108 (1960).

10) For the definition, see [2].

11) For the proof, see [2] or [3].

12) This property may be understood as the superharmonicity of the strong (or global) sense.