224. *On Imbeddings and Colorings of Graphs. II*

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§ 1. Introduction. In this paper we use the same definitions and notations as in the part I. The following theorem is proved in this paper.

Theorem (1.1). *If there exists an m-imbeddable and n-chromatic graph, there exists a graph of which genus is m′ and the chromatic number is n′, for any m′ and n′ satisfying* $m'\geqq m$ *and* $2\leqq n'\leqq n$.

In § 2 we prove theorem (2.5), which is more general than theorem (1.1), by using part I. But theorem (1.1) can be proved directly by the idea which professor Y. Saito told me. An outline of this idea is shown at the end of this paper.

Lemma (2.1). *Let the chromatic number of H be larger than 2. If there is an imbedding of H into M, there is a graph G such that (i) H and G have the same chromatic number, (ii) G has no k-circuit for* $k<3$, *and (iii) there is a 2-cell imbedding G(M).*

Proof. Let $H(M)$ be the given imbedding, and let α be any component of M-$H(M)$. If H is n-chromatic, there is a color-classification $H^0=\gamma_1\cup\cdots\cup\gamma_n$.

We can take arcs $a_1,\cdots,a_p$ in α joining the vertices in $\mathrm{Bd}\,\alpha$ and not intersecting each other such that any component of $\alpha-a_1\cup\cdots\cup a_p$ is an open 2-cell. Let A_i, B_i be the edges of a_i. Here we permit happening $A_i=B_i$. Let C_i be the center of a_i. Then, we construct as follows an n-chromatic graph H_α imbedding $H_\alpha(M)$ and color-classes $\gamma_{\alpha,1},\cdots,\gamma_{\alpha,n}$:

(i) $H^0_\alpha=H^0\cup\{C_1,\cdots,C_p\}$,
$H^1_\alpha=H^1\cup\{(A_i,c_i),(B_i,C_i)\mid i=1,\cdots,p\}$.

(ii) $H_\alpha(M)\mid H=H(M)$, $H(M)(C_i)=C_i$ and $H_\alpha(M)\mid(A_i,C_i)\cup(B_i,C_i)$ is onto a_i.

(iii) As the chromatic number of $H\geqq 3$, for a_i we can fix γ_j such that $\gamma_j\neq\gamma(A_i),\gamma(B_i)$. If we note this γ_j by $\gamma(a_i)$, we have the color-classes of H_α as follows:
$$\gamma_{\alpha,j}=\gamma_j\cup\{C_i\mid\gamma(a_i)=\gamma_j\},\ (j=1,\cdots,n).$$

We repeat the same modification for all non cellular components of $M-H(M)$, and finally we obtain H_1, $H_1(M)$ and color-classification $H^0=\gamma_{1,1}\cup\cdots\cup\gamma_{1,n}$. H_1 satisfies the conditions (i) and (iii) which G is to satisfy.

Next, let $a_1,\cdots,a_q$ be all the 1-circuits contained in H_1 and let A_i be the vertex on a_i, namely $a_i=(A_i,A_i)\in H^1_1$. To take away

1-circuits from H_1, we construct H_2 and the color-classification $H_2^0=\gamma_{2,1}\cup\cdots\cup\gamma_{2,n}$ as follows:

(i) $H_2^0=H_1^0\cup\{C_1,\cdots,C_q\}$,
$H_2^1=(H_1^1-\{a_1,\cdots,a_q\})\cup\{(A_i,C_i)_0,(C_i,A_i)_1\mid i=1,\cdots,q\}$.

(ii) For a_i, fix $\gamma_{1,j}$ such that $\gamma_{1,j}\neq\gamma_1(A_i)$ and note it by $\gamma_1(a_i)$, then we give the color-classes of H_2 by
$$\gamma_{2,j}=\gamma_{1,j}\cup\{C_i\mid\gamma_1(a_i)=\gamma_{1,j}\},\quad j=1,\cdots,n.$$

Clearly, H_2 has no 1-circuit, and we can see that H_2 is n-chromatic by the same discussion as (4) in the proof of (3.2) in the part I.

Next, we construct H_3 which contains no 2-circuit. Let $(A_i,B_i)_j$, $i=1,\cdots,r$, $j=0,\cdots,m_i(m_i\geqq 1)$, be all the arcs which are contained some 2-circuits in H_2.

Then, we construct H_3 and the color-classification $H_3^0=\gamma_{3,1}\cup\cdots\cup\gamma_{3,n}$ as follows:

(i) $H_3^0=H_2^0\cup\{C_{i,j}\mid i=1,\cdots,r,j=1,\cdots,m_i\}$,
$H_3^1=(H_2^1-\{(A_i,B_i)_j\mid i=1,\cdots,r,j=1,\cdots,m_i\})$
$\cup\{(A_i,C_i)_j,(C_i,B_i)_j\mid i=1,\cdots,r,j=1,\cdots,m_i\}$.

(ii) For (A_i,B_j) fix $\gamma_{2,j}$ such that $\gamma_{2,j}\neq\gamma_2(A_i),\gamma_2(B_i)$, and note it by $\gamma_2(A_i,B_i)$, then we give the color-classes of H_3 by
$\gamma_{3,k}=\gamma_{2,k}\cup\{C_{i,j}\mid\gamma_2(A_i,B_i)=\gamma_{2,k},j=1,\cdots,m_i\}$, $(k=1,\cdots,n)$.

Clearly H_3 does not contain 1-circuit nor 2-circuit, and we can see that H_3 is n-chromatic by the same discussion as (4) in the proof of (3.2) in the part I.

$H_3=G$ is the required graph: as there is the natural homeomorphism of G onto H_1 as 1-complexes, $G(M)$ can be naturally given from $H_1(M)$, and this is a 2-cell imbedding. This proves (2.1).

Lemma (2.2). *If G is n-chromatic, there is a subgraph H of G with chromatic number m for any m with $1\leqq m\leqq n$.*

Proof. There is a color-classification $G^0=\gamma_1\cup\cdots\cup\gamma_n$. And give H by
$$H^0=\gamma_1\cup\cdots\cup\gamma_m,$$
$$H^1=\{(A,B)\mid A,B\in H^0,(A,B)\in G^1\}.$$
By the definition of chromatic number, we can easily prove that H is m-chromatic. This proves (2.2).

From (3.7) in the part I, (2.1) and (2.2), we have,

Lemma (2.3). *Let H be an n-chromatic graph, $n\geqq 3$ and have an imbedding into surface M. Then, for any m with $n\geqq m\geqq 3$, there is a graph G which is m-chromatic and has simplest imbedding into M.*

Lemma (2.4). *For any surface M_j there is a graph which is 2-chromatic and has a simplest imbedding into M.*

Proof. There exist a simplicial (triangulate) division K of M, i.e. K is a finite simplical complex and $|K|=M$.

We construct the required graph G and the simplest imbedding $G(M)$ from K. Let $A(\alpha)_1, A(\alpha)_2$, and $A(\alpha)_3$ be the vertices of any 2-simplex of K. (We note $A(\alpha)$ simply A hereafter.) Let $A_{i,j}$ be the barycenter of an 1-simplex (A_i, A_j), $i=1, 2, 3$, $i<j$, and let A_{123} be the barycenter of a 2-simplex α. And let $A_{i,ij}$, $A_{j,ij}$, and $A_{i,123}$ be the barycenters respectively of (A_i, A_{ij}), (A_j, A_{ij}), and (A_i, A_{123}) which are the 1-simplexes of the barycentric subdivision of K, $i, j=1, 2, 3$, $i<j$.

As a subcomplex of 1-skeleton of second barycentric subdivision of K, we have a graph $G=(G^0, G^1)$ as follows. Let

$$U_1=\{A_i \mid \alpha \in K, i=1, 2, 3\}$$
$$U_2=\{A_{123} \mid \alpha \in K\},$$
$$U_3=\{A_{i,ij}, A_{j,ij} \mid \alpha \in K, i, j=1, 2, 3, \quad i<j\},$$
$$U_4=\{A_{ij} \mid \alpha \in K, \quad i, j=1, 2, 3, \quad i<j\},$$
$$U_5=\{A_{i,123} \mid \alpha \in K, \quad i=1, 2, 3\},$$

where A means $A(\alpha)$ and α is a 2-simplex in K. Then, G is given by

$$G^0=U_1\cup U_2\cup U_3\cup U_4\cup U_5$$
$$G^1=\{(A_i, A_{i,123}), (A_{i,123}, A_{123}) \mid \alpha \in K, i=1, 2, 3\}$$
$$\cup\{(A_{123}, A_{ij}), (A_{i,ij}, A_{ij}), (A_{j,ij}, A_{ij}) \mid \alpha \in K,$$
$$i, j=1,2,3, \quad i<j\}.$$

We have naturally $G(M)$ from second barycentric subdivision of K. Then, $G(M)$ is a 4-gon imbedding. And we can see that G has no k-circuit with $k<4$. Therefore by (2.1) in the part I, $G(M)$ is a simplest imbedding.

As $U_1\cup U_2\cup U_3$ and $U_4\cup U_5$ are SC-sets, G is 2-colorable. This proves (2.4).

From (2.3) and (2.4) we have,

Theorem (2.5). *If there exists an imbedding of an n-chromatic graph into a surface M, there is an m-chromatic graph having simplest imbedding into M for any m with $2\leqq m\leqq n$.*

When we restrict theorem (2.5) in orientable case (i.e. surfaces are assumed to be orientable, we have (1.1). But we can prove (1.1), without using theorem (2.1) in the part I. We show the outline of the proof:

Let $I=(I^0, I^1)$ be the graph such that

$$I^0=U\cup V,$$

where both U and V have 3 elements and $U\cap V=\phi$

$$I^1=\{(A, B) \mid A\in U, B\in V\}.$$

Then, I is 2-chromatic and has genus 1 (c.f. [1]).

Let G be n-chromatic ($m\geqq 2$) and have an imbedding into an orientable manifold of genus k. By (2.2) we have m-chromatic subgraph H of G for any m with $2\leqq m\leqq n$. Let h be the genus of H. Atach H and I by one arc and the two vertices which are ends of the arc, then the resulting graph $H \cup I$ has genus $k+1$ and chromatic number m. Then, by attaching $(k-h)I$ to H, we have a graph which chromatic number is m and genus k.

References

[1] C. Kuratowski: Sur le problème des courbes gauches en topologie. Fund. Math., **15**, 271-283 (1963).

[2] J. R. Munkres: Elementary differential topology. Amer. Math. Studies, **54** (1963). Princeton University Press.

[3] Oystein Ore: Theory of graphs. Amer. Math. Soc. Colloquium Publications Vol. XXXVIII (1962).

[4] J. W. T. Youngs: Minimal imbeddings and the genus of a graph. J. Math. Meck., **12**, 303-315 (1963).