140. The Structure of Quasi-Minimal Sets

By Shigeo KONO
Department of Mathematics, Josai University

(Comm. by Kinjirô KUNUGI, M. J. A., June 12, 1970)

1. Introduction. The concept of the quasi-minimal sets, introduced by H. F. Hilmy [1], plays rather important roles for the investigation of the structure of the center of the compact dynamical systems.

In this paper, we study mainly the three problems, i.e., (a) how a quasi-minimal set contains minimal sets, (b) the qualities of these minimal sets, (c) the behaviors of the orbits near these minimal sets. Main results obtained are as follows:

Theorems 9 and 10 for (a),

Theorems 8, 12 and 13 for (b), and

Theorem 14 for (c).

2. Definitions and notations.

X: a compact metric space.

R: a real line.

π: $X\times R\to X$ is a mapping which satisfies

1) $\pi \in C[X\times R]$,

2) $\pi(x, 0)=x$, and

3) $\pi(\pi(x, s), t)=\pi(x, s+t)$.

The triple (X, R, π) is a compact dynamical system whose phase space, phase group, and phase projection are X, R, and π, respectively.

$\gamma(x)=\{\pi(x, t)\,;\, t\in R\}$ is the orbit passing through $x\in X$.

$\gamma^+(x)=\{\pi(x, t)\,;\, t\geqq 0\}$ and $\gamma^-(x)=\{\pi(x, t)\,;\, t\leqq 0\}$ are respectively positive semi-orbit and negative semi-orbit from $x\in X$.

$\Lambda^+(x)=\bigcap_{0\leqq t}\overline{\gamma^+(\pi(x, t))}$ and $\Lambda^-(x)=\bigcap_{0\geqq t}\overline{\gamma^-(\pi(x, t))}$ are the positive and negative limit set of $\gamma(x)$, respectively.

$\gamma(x)$ is positively (negatively) Poisson stable if and only if $\Lambda^+(x)\cap\gamma(x)\neq\phi$ ($\Lambda^-(x)\cap\gamma(x)\neq\phi$).

$\gamma(x)$ is Poisson stable if and only if it is both positively and negatively Poisson stable.

$\gamma(x)$ is positively (negatively) asymptotic if and only if $\gamma(x)\cap\Lambda^+(x)=\phi$ and $\Lambda^+(x)\neq\phi$ ($\gamma(x)\cap\Lambda^-(x)=\phi$ and $\Lambda^-(x)\neq\phi$).

A subset S of X is invariant if and only if $\gamma(x)\subset S$ holds for any $x\in S$.

A closed and invariant set F is minimal if and only if it contains no proper subsets which are closed and invariant.

3. The structure of S_π, S_μ and S_σ.

Lemma 1. *$\gamma(x)$ is Poisson stable if and only if*

$$\Lambda^+(x)=\Lambda^-(x)=\overline{\gamma(x)}$$

holds.

The proof of this lemma is easy.

Definition 2. *We call a set S quasi-minimal if there exists a point x of S such that $\gamma(x)$ is Poisson stable and is everywhere dense in S, i.e., $S=\overline{\gamma(x)}$.*

Definition 3 [2]. *Let S be a quasi-minimal set.*

1) *A point x of S is called π-point if $\overline{\gamma^+(x)}=\overline{\gamma^-(x)}=S$.*

2) *A point x of S is called μ-point if* (a) *$\overline{\gamma^+(x)}=S$ and $\overline{\gamma^-(x)}\subsetneqq S$ or* (b) *$\overline{\gamma^+(x)}\subsetneqq S$ and $\overline{\gamma^-(x)}=S$ holds.*

3) *A point x of S is called σ-point if $\overline{\gamma^+(x)}\subsetneqq S$ and $\overline{\gamma^-(x)}\subsetneqq S$.*

Definition 4. *Let S be a quasi-minimal set. We define S_π, S_μ and S_σ as follows:*

$S_\pi=\{x\,;\ x$ *is a π-point of* $S\}$,

$S_\mu=\{x\,;\ x$ *is a μ-point of* $S\}$,

$S_\sigma=\{x\,;\ x$ *is a σ-point of* $S\}$.

It is known that S_π, S_μ and S_σ are all invariant [2].

The following Proposition 4 is found in T. Saito's paper [3].

Proposition 4. *Let S be a quasi-minimal set.*

1) *If $x\in S_\pi$, then $\gamma(x)$ is Poisson stable.*

2) *if $x\in S_\mu$, then $\gamma(x)$ is*

a) *positively Poisson stable and negatively asymptotic, or*

b) *negatively Poisson stable and positively asymptotic.*

Here we consider the problem whether the inverse of proposition 4–1) holds or not. The answer to the problem is as follows:

Proposition 5. *Let S be a quasi-minimal set.*

$$x\in S_\pi \Longleftrightarrow \begin{cases} 1) & \gamma(x) \text{ is Poisson stable.} \\ 2) & \overline{\gamma(x)}=S. \end{cases}$$

Proof ($\Rightarrow$). We know that $\overline{\gamma(x)}=S$, because $\overline{\gamma^+(x)}=\overline{\gamma^-(x)}=S$. This fact and Proposition 4–1) completes the proof of ($\Rightarrow$).

($\Leftarrow$). We know from the assumption 1) and Lemma 1 that

$$\overline{\gamma(x)}=\Lambda^+(x)=\Lambda^-(x).$$

On the other hand

$$\Lambda^+(x)\subset\overline{\gamma^+(x)}\subset S,$$

and

$$\Lambda^-(x)\subset\overline{\gamma^-(x)}\subset S$$

holds. These facts and the assumption 2) imply

$$S=\overline{\gamma^+(x)}=\overline{\gamma^-(x)}.$$

Thus $x\in S_\pi$. Q.E.D.

Next we give the necessary and sufficient conditions from a standpoint of limit sets for a point of a quasi-minimal set to be π-, or μ-, or σ-points:

Proposition 6.

1) $x \in S_\pi \Longleftrightarrow \Lambda^+(x)=\Lambda^-(x)=S.$

2) $x \in S_\mu \Longleftrightarrow$ 1) $\Lambda^+(x)=S$ *and* $\Lambda^-(x)\subsetneqq S,$
or 2) $\Lambda^+(x)\subsetneqq S$ *and* $\Lambda^-(x)=S.$

3) $x \in S_\sigma \Longleftrightarrow \Lambda^+(x)\subsetneqq S$ *and* $\Lambda^-(x)\subsetneqq S.$

Proof. 1) can be easily proved using Lemma 1 and Proposition 5.

2) Let x be a point of S_μ. Then, a) $\overline{\gamma^+(x)}=S$ and $\overline{\gamma^-(x)}\subsetneqq S$, or b) $\overline{\gamma^+(x)}\subsetneqq S$ and $\overline{\gamma^-(x)}=S$. We shall prove only the case a). The case b) can be proved similarly. In the case a), $\gamma(x)$ is positively Poisson stable and negatively asymptotic, so that

$$\phi \neq \Lambda^-(x)\subsetneqq \Lambda^+(x)=\overline{\gamma(x)}.$$

The closedness and invariantness of S imply that

$$S=\overline{\gamma^+(x)}\subset\overline{\gamma(x)}\subset S,$$

which means that $\overline{\gamma(x)}=S$. Thus we know that

$$\Lambda^+(x)=S \quad \text{and} \quad \Lambda^-(x)\subsetneqq S. \tag{1}$$

Conversely, let us assume that there exists a point x of S which satisfies (1). Then

$$S=\Lambda^+(x)\subset\overline{\gamma^+(x)}\subset S,$$

so that

$$\overline{\gamma^+(x)}=S. \tag{2}$$

But $\overline{\gamma^-(x)}\subsetneqq S$, for if $\overline{\gamma^-(x)}=S$, then $x \in S_\pi$, so that $\Lambda^-(x)=S$ by Proposition 6–1), which contradicts the assumption (1). Thus $x \in S_\mu$.

3) The fact that 1), 2) and 3) are mutually exclusive proves 3).

Q.E.D.

Proposition 4 tells us the nature of the orbits in S_π and S_μ. Now we study structure of S_σ.

Proposition 7. $x \in S_\sigma \Rightarrow \overline{\gamma(x)}\subset S_\sigma$.

Proof. As S_σ is invariant [2], $\gamma(x)\subset S_\sigma$ for all $x \in S_\sigma$. Let x be a point of S_σ.
Then $(\forall y \in \Lambda^+(x))$

$$\begin{cases}\Lambda^+(y)\subset\overline{\gamma(y)}\subset\Lambda^+(x)\subsetneqq S \quad \text{and}\\ \Lambda^-(y)\subset\overline{\gamma(y)}\subset\Lambda^+(x)\subsetneqq S.\end{cases}$$

These facts imply that $y \in S_\sigma$. Thus $\Lambda^+(x)\subset S_\sigma$. We can prove similarly that $\Lambda^-(x)\subset S_\sigma$. Therefore

$$\overline{\gamma(x)}=\gamma(x)\cup\Lambda^+(x)\cup\Lambda^-(x)\subset S_\sigma.$$

Q.E.D.

Theorem 8. *The open kernel of* S_σ *is empty.*

Proof. If $S_\sigma=\phi$, Theorem 8 is trivial. Let us assume that $S_\sigma\neq\phi$. As S_π is invariant [2],

$$\gamma(x)\subset S_\pi\subset S \tag{1}$$

for any $x\in S_\pi$. On the other hand, if $x\in S_\pi$, then by Proposition 5

$$\overline{\gamma(x)}=S. \tag{2}$$

We know by (1) and (2)

$$\overline{S_\pi}=S,$$

that is, S_π is everywhere dense in S. Therefore, for any point x of S_σ and for any neighborhood $U(x)$ of this x, $U(x)\cap S_\pi\neq\phi$. Thus no points of S_σ are interior points, so that the open kernel of S_σ is empty. Q.E.D.

A quasi-minimal set S is compact and invariant, so S contains at least one minimal set [2]. But it is an important problem that in what way S contains minimal sets. We give the answer to this problem as follows.

Theorem 9. *A quasi-minimal set S is minimal if and only if $S=S_\pi$.*

Proof. If S is minimal, then $\gamma(x)$ is Poisson stable and $\overline{\gamma(x)}=S$ for any $x\in S$. This means that if $x\in S$, then $x\in S_\pi$. Thus $S\subset S_\pi$. But, of course $S_\pi\subset S$. Therefore $S=S_\pi$. Conversely, let us assume that $S=S_\pi$. If S is not minimal, then S contains a minimal set M. $M\subset S_\pi$ implies that $\overline{\gamma(x)}=S$ for all $x\in M$ by Proposition 5. But $\overline{\gamma(x)}=M$ for all $x\in M$, because M is closed and invariant. Further $M\subsetneqq S$. Thus we arrive at a contradiction. Therefore S is minimal. Q.E.D.

Corollary 9.1. *A quasi-minimal set S is not minimal if and only if $S_\mu\cup S_\sigma\neq\phi$.*

Theorem 10. *If a quasi-minimal set S is not minimal, then S_σ contains all minimal sets contained in S.*

Proof. Let M be a minimal set contained in S. For any $y\in M$, $\overline{\gamma(y)}=M\subsetneqq S$. This shows that $M\cap S_\pi=\phi$ (Proposition 5). Thus $M\subset S_\mu\cup S_\sigma$. Now we assume that $M\cap S_\mu\neq\phi$. For any point $x\in M\cap S_\mu$ one of the following two cases holds:

a) $\overline{\gamma^+(x)}=S$ and $\overline{\gamma^-(x)}\subsetneqq S$,

b) $\overline{\gamma^+(x)}\subsetneqq S$ and $\overline{\gamma^-(x)}=S$.

The case a), however, never occurs because it contradicts the fact that $\overline{\gamma^+(x)}\subset\overline{\gamma(x)}\subset M\subsetneqq S$. Also the case b) never occurs because of the similar reason as in the case a). Thus we know $M\cap S_\mu=\phi$, which implies that $M\subset S_\sigma$. Q.E.D.

Corollary 10.1. *A quasi-minimal set S is not minimal if and only if $S_\sigma\neq\phi$.*

If a quasi-minimal set S is not minimal, then S_σ contains all minimal sets contained in S. Here we study the behaviors of orbits near the minimal sets.

For this purpose, we first give some definitions and notations.

U is an arbitrary neighborhood of a minimal set of the dynamical system (X, R, π). We classify $\bar{U}\backslash F$ as follows:

$$N_U^+=\{x\,;\, x\in \bar{U}\backslash F,\ C^+(x)\subset \bar{U}\},$$
$$N_{\bar{U}}^-=\{x\,;\, x\in \bar{U}\backslash F,\ C^-(x)\subset \bar{U}\},$$
$$G_U=\{x\,;\, x\in \bar{U}\backslash F,\ C^+(x)\not\subset \bar{U},\ C^-(x)\not\subset \bar{U}\},$$
$$N_U=N_U^+\cap N_{\bar{U}}^-.$$

Definition 10. *We call a minimal set F isolated, if there exists a neighborhood of F which contains no minimal sets other than F.*

Definition 11 [4]. *An isolated minimal set F is called a saddle minimal set, if there exists a neighborhood U of F such that $\bar{G}_U\cap F\neq\phi$.*

Theorem 12. *Let S be a quasi- minimal set which is not minimal.*
1) *All isolated minimal sets contained in S are saddle minimal sets.*
2) *If S contains a minimal set which is not isolated, then S contains infinitely many minimal sets.*
3) *If S contains a finite number of minimal sets, then these minimal sets are all saddle minimal sets.*

Proof. 1) S is compact, invariant, and not minimal, so there exists a compact minimal set which is a proper subset of S. But it is known that if a proper subset F of S is an isolated minimal subset, then F is a saddle minimal set [4].
2) is directly proved by Definition 10.
3) is proved by Theorem 12–1) and the fact that all minimal sets contained in S are isolated in this case. Q.E.D.

Theorem 13. *If a quasi-minimal set S is not minimal, then all the minimal sets contained in S are nowhere dense.*

Proof. The open kernels of minimal sets contained in S are subsets of S_σ (Theorem 10), so that these open kernels are all empty (Theorem 8). This result completes the proof because minimal sets are closed. Q.E.D.

The behaviors of orbits in S_π and S_μ are known (Proposition 4). But it remains an open problem to determine the behaviors of orbits in S_σ, as far as I know. The following Theorem 14 is a result of an attempt to solve this problem.

Let F be an isolated minimal set contained in S_σ. There exists an open neighborhood U of F, which contains no minimal sets other than F.

$V=S_\sigma\cap U$ is a relative neighborhood of F in S_σ. Let $\tilde{V}$ be the relative closure of V in S_σ. Then

$$\begin{aligned}\tilde{V}\backslash F&=(S_\sigma\cap \bar{U})\backslash F\\ &=(S_\sigma\cap N_{\bar{U}}^+)\cup(S_\sigma\cap N_{\bar{U}}^-)\cup(S_\sigma\cap G_U),\end{aligned}$$

because $\bar{U}\backslash F = N_U^+ \cup N_U^- \cup G_U$. Here we take n_V^+, n_V^-, n_V, g_V as follows:

$$n_V^+ = S_\sigma \cap N_U^+,$$
$$n_V^- = S_\sigma \cap N_U^-,$$
$$n_V = n_V^+ \cap n_V^-,$$
$$g_V = S_\sigma \cap G_U.$$

Then $\tilde{V}\backslash F = n_V^+ \cup n_V^- \cup g_V$.

The following facts are valid by the compactness of $\bar{U}$ [4]:

1) if $x \in N_U^+ \backslash N_U$, then $\gamma(x)$ is positively asymptotic and $\gamma^-(x) \cap (X\backslash \bar{U}) \neq \phi$.

2) if $x \in N_U^- \backslash N_U$, then $\gamma(x)$ is negatively asymptotic and $\gamma^+(x) \cap (X\backslash \bar{U}) \neq \phi$.

Therefore, if $x \in n_V^+ \backslash n_V$, then $x \in (N_U^+ \backslash N_U) \cap S_\sigma$, so that $\gamma(x)$ is positively asymptotic. On the other hand,

$$[\gamma^-(x) \cap (X\backslash \bar{U})] \cap S_\sigma = \gamma^-(x) \cap (S_\sigma \backslash \tilde{V}), \quad \text{while}$$
$$\gamma^-(x) \cap (X\backslash \bar{U}) \cap S_\sigma = [\gamma^-(x) \cap S_\sigma] \cap (X\backslash \bar{U})$$
$$= \gamma^-(x) \cap (X\backslash \bar{U}) \neq \phi, \quad \text{therefore} \quad \gamma^-(x) \cap (S_\sigma \backslash \tilde{V}) \neq \phi.$$

Similarly, if $x \in n_V^- \backslash n_V$, then $\gamma(x)$ is negatively asymptotic and $\gamma^+(x) \cap (S_\sigma \backslash \tilde{V}) \neq \phi$.

The following two propositions are clear:

1) if $x \in n_V$, then $\gamma(x) \subset \tilde{V}\backslash F$,

2) if $x \in g_V$, then $\gamma^+(x) \cap (S_\sigma \backslash \tilde{V}) \neq \phi$ and $\gamma^-(x) \cap (S_\sigma \backslash \tilde{V}) \neq \phi$.

Finally, it is known that if $x \in F$, then $\gamma(x) \subset F$ and $\gamma(x)$ is Poisson stable.

We summarize above results as follows:

Theorem 14. *Let S be a quasi-minimal set which is not minimal. Let F be an isolated minimal set contained in S.*

Then, there exists a relative neighborhood V such that the orbits passing a point of $\tilde{V}$, *the relative closure of V in* S_σ, *are classified as follows*:

1) *if* $x \in F$, *then* $\gamma(x)$ *is Poisson stable and* $\gamma(x) \subset F$,

2) *if* $x \in n_V^+ \backslash n_V$, *then* $\gamma(x)$ *is positively asymptotic and* $\gamma^-(x) \cap (S_\sigma \backslash \tilde{V}) \neq \phi$.

3) *if* $x \in n_V^- \backslash n_V$, *then* $\gamma(x)$ *is negatively asymptotic and* $\gamma^+(x) \cap (S_\sigma \backslash \tilde{V}) \neq \phi$.

4) *if* $x \in n_V$, *then* $\gamma(x) \subset \tilde{V}\backslash F$.

5) *if* $x \in g_V$, *then* $\gamma^+(x) \cap (S_\sigma \backslash \tilde{V}) \neq \phi$ *and* $\gamma^-(x) \cap (S_\sigma \backslash \tilde{V}) \neq \phi$.

References

[1] H. F. Hilmy: Sur les ensembles quasi-minimaux dans les Systemès dynamiques. Ann. Math., **37** (1936).

[2] T. M. Cherry: Topological Properties of the solutions of ordinary differential equations. Am. Jour. of Math., Vol. 59 (1937).

[3] T. Saito: Global theory of the dynamical systems (in Japanese). Funk. Ekv. XVII N-ro 1 (1964).

[4] ——: Isolated Minimal Sets. Funkcialaj Ekvacioj, **11** (1968).