252. *On Wiener Functions of Order m*

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1. Let Ω be an open subset of the n-dimensional Euclidian space $R^n(n\geq 2)$ and f_i be a continuous function on the boundary of $\Omega(1\leq i\leq m)$. Riquier's problem for a polyharmonic equation $\Delta^m u=0$ on Ω is to find a function u such that $\Delta^m u=0$ in Ω and $(-\Delta)^{i-1}u=f_i$ on the boundary of Ω for each $i(1\leq i\leq m)$.

For a unit disk it was solved by Riquier and for a bounded open set by M. Itô [2].

In this note we shall show that for an unbounded open subset Ω its problem can be solved by means of Wiener ideal boundary Δ_W and Wiener harmonic boundary Γ_W of Ω (Theorem 3).

Let f_i be a continuous function on $\Delta_W(1\leq i\leq m)$. Then there exists a function $h_{(f_1,f_2,\dots,f_m)}$ on Ω such that

$$\Delta^m h_{(f_1,f_2,\dots,f_m)}=0$$

in Ω and for each $i\ (1\leq i\leq m)$, on Γ_W

$$(-\Delta)^{i-1}h_{(f_1,f_2,\dots,f_m)}=f_i$$

if and only if Ω satisfies the condition

$$\int G_\Omega^{(m-1)}(x,y)dy<+\infty$$

for some point x in Ω, where G_Ω being the Green function of Ω,

$$G_\Omega^{(m-1)}(x,y)=\int\cdots\int G_\Omega(x,z_1)G_\Omega(z_1,z_2)\cdots G_\Omega(z_{m-2},y)dz_1dz_2\cdots dz_{m-2}.$$

2. Let Ω be an open subset of R^n. We call a real valued function u in the class $C^{2m}(\Omega)$ is polyharmonic of order m in Ω if we have in Ω

$$\Delta^m u=\left(\sum_{k=1}^{n}\frac{\partial^2}{\partial x_k^2}\right)^m u=0.$$

For the Green function G_Ω of Ω and an integer $i\geq 1$, we put

$$G_\Omega^{(i)}(x,y)=\int\cdots\int G_\Omega(x,z_1)G_\Omega(z_1,z_2)\cdots G_\Omega(z_{i-1},y)dz_1dz_2\cdots dz_{i-1}.$$

By a suitable normalization we have $(-\Delta_y)^i G_\Omega^{(i)}(x,y)=\varepsilon_x$ in Ω, where ε_x is the Dirac measure at x.

From now on, let $m(\geq 1)$ be a fixed integer and i be any integer $1\leq i\leq m$. As to the solution of Riquier's problem, M. Itô [2] proved

Lemma 1. *Let Ω be a bounded open subset of R^n and $(f_i)_{i=1}^m$ be a system of bounded continuous functions on the boundary $\partial\Omega$ of Ω. Then there exist m positive Radon measures $\varepsilon_{x,c\Omega}^{(i)}(1\leq i\leq m)$ on $\partial\Omega$, and the function*

$$H_{\Omega}(x;(f_i)_{i=1}^m)=\sum_{i=1}^m\int f_i(y)d\varepsilon_{x,c\Omega}^{(i)}(y)$$

is polyharmonic of order m in Ω.

Remark 1. The above system $(\varepsilon_{x,c\Omega}^{(i)})_{i=1}^m$ of balayaged measures of $(\varepsilon_x, 0, \cdots, 0)$ on $\partial\Omega$ satisfies

$$\int f(y)d\varepsilon_{x,c\Omega}^{(1)}(y)=H_f^{\Omega}(x)$$

and for each $i (1\leq i\leq m)$

$$\int f(y)d\varepsilon_{x,c\Omega}^{(i)}(y)=\int H_f^{\Omega}(z)G_{\Omega}^{(i-1)}(x,z)dz$$

for any bounded continuous function f on $\partial\Omega$, where H_f^{Ω} is the solution of the Dirichlet problem in Ω for the boundary value f.

Let Ω be an open subset of R^n and $\{\Omega_k\}_{k=1}^\infty$ be an exhaustion of Ω namely a sequence of relatively compact open subsets of Ω such that $\bar{\Omega}_k\subset\Omega_{k+1}$ and $\bigcup_{k=1}^\infty\Omega_k=\Omega$. Then by Lemma 1 we have a sequence $(\varepsilon_{x,c\Omega_k}^{(i)})_{k=1}^\infty$ of positive Radon measures for each $i(1\leq i\leq m)$.

Now we shall define Wiener functions of order i as follows.

Definition. We shall call a bounded continuous function f on Ω a Wiener function of order i if for any exhaustion $\{\Omega_k\}_{k=1}^\infty$ of Ω the sequence $\left\{\int f(y)d\varepsilon_{x,c\Omega_k}^{(i)}(y)\right\}_{k=1}^\infty$ converges at each point x in Ω.

We denote by $W^{(i)}(\Omega)$ the totality of Wiener functions of order i on Ω and for f in $W^{(i)}(\Omega)$ we put

$$h_f^{(i)}(x)=\lim_{k\to\infty}\int f(y)d\varepsilon_{x,c\Omega_k}^{(i)}(y)$$

and $W_0^{(i)}(\Omega)=\{f\in W^{(i)}(\Omega);\ h_f^{(i)}=0 \text{ on } \Omega\}$ $(1\leq i\leq m)$.

We note that a function in $W^{(1)}(\Omega)$ (or $W_0^{(1)}(\Omega)$) is a usual Wiener function (or Wiener potential) on Ω (cf. [1]).

For brevity we say that Ω satisfies the condition [i] if

$$\int G_{\Omega}^{(i-1)}(x,y)dy<+\infty$$

for some point x in Ω. By the condition [i] we have the finiteness of the above integral for any point x in Ω.

As to the class $W^{(i)}(\Omega)$ we shall show

Theorem 1. *Let Ω be an open subset of R^n. Then we have $W^{(i)}(\Omega)=W^{(1)}(\Omega)$ and $W_0^{(i)}(\Omega)=W_0^{(1)}(\Omega)$ if and only if Ω satisfies the condition* [i].

Proof. Let f be a bounded continuous function on Ω and $\{\Omega_k\}_{k=1}^\infty$ be an exhaustion of Ω. By Remark 1, we have

$$\int\left(\int f(y)d\varepsilon_{x,c\Omega_k}^{(1)}(y)\right)G_{\Omega_k}^{(i-1)}(x,z)dx=\int f(y)d\varepsilon_{z,c\Omega_k}^{(i)}(y),$$

where G_{Ω_k} is the Green function of Ω_k. If f is in $W^{(1)}(\Omega)$ then for each point x in Ω, $\lim_{k\to\infty}\int f(y)d\varepsilon_{x,c\Omega_k}^{(1)}(y)=h_f^{(1)}(x)$.

By the condition [i], for each point z in Ω

$$\left|\int\left(\int f(y)\,d\varepsilon^{(1)}_{x,c\Omega_k}(y)\right)G^{(i-1)}_{\Omega_k}(x,z)dx\right|\leqslant\sup_{x\in\Omega}|f(x)|\cdot\int G^{(i-1)}_{\Omega}(z,y)dy<+\infty,$$

therefore we know $\lim_{k\to\infty}\int f(y)d\varepsilon^{(i)}_{z,c\Omega_k}(y)=\int h^{(1)}_f(x)G^{(i-1)}_{\Omega}(x,z)dx$ and so f is in $W^{(i)}(\Omega)$. Hence we have $W^{(1)}(\Omega)\subset W^{(i)}(\Omega)$ and similarly $W^{(1)}_0(\Omega)\subset W^{(i)}_0(\Omega)$.

We shall show the inverse inclusions $W^{(i)}(\Omega)\subset W^{(1)}(\Omega)$ and $W^{(i)}_0(\Omega)\subset W^{(1)}_0(\Omega)$. Its proof is suggested by M. Itô.

Let x_0 be a fixed point in Ω and φ_{x_0} be a non-negative $C^{(i-1)}$-function with compact support which is invariant under rotation around x_0 and $\int\varphi_{x_0}(y)dy=1$.

If f is in $W^{(i)}(\Omega)$, then the sequence $\left\{\int H^{\Omega_k}_f(x)G^{(i-1)}_{\Omega_k}(x,z)dx\right\}_{k=1}^{\infty}$ converges at each point z in Ω, so by the condition [i] the sequence $\left\{\int H^{\Omega_k}_f(x)\left(\int G^{(i-1)}_{\Omega_k}(x,z)(-\Delta)^{i-1}\varphi_{x_0}(z)dz\right)dx\right\}_{k=1}^{\infty}$ converges.

Since $\varphi_{x_0}(x)=\int G^{(i-1)}_{\Omega_k}(x,z)(-\Delta)^{i-1}\varphi_{x_0}(z)dz$ for sufficient large k, we have

$$\int H^{\Omega_k}_f(x)\left(\int G^{(i-1)}_{\Omega_k}(x,z)(-\Delta)^{i-1}\varphi_{x_0}(z)dz\right)dx$$
$$=\int H^{\Omega_k}_f(x)\varphi_{x_0}(x)dx=H^{\Omega_k}_f(x_0).$$

Hence the sequence $\left\{\int f(y)d\varepsilon^{(1)}_{x,c\Omega_k}(y)\right\}_{k=1}^{\infty}$ converges at a point x_0 and so f is in $W^{(1)}(\Omega)$. Similarly we have $W^{(i)}_0(\Omega)\subset W^{(1)}_0(\Omega)$.

Conversely if $W^{(i)}(\Omega)=W^{(1)}(\Omega)$, 1 being in $W^{(i)}(\Omega)$, we have the condition [i].

By Theorem 1, we have

Corollary 1. *The condition* [m] *implies*
$W^{(1)}(\Omega)=W^{(2)}(\Omega)=\cdots=W^{(m)}(\Omega)$ *and* $W^{(1)}_0(\Omega)=W^{(2)}_0(\Omega)=\cdots=W^{(m)}_0(\Omega)$.

As to a function $h^{(i)}_f$ we shall show two lemmas.

Lemma 2. *If Ω satisfies the condition* [i] *and f is in $W^{(1)}(\Omega)$ then we have $\Delta^i h^{(i)}_f=0$ in Ω.*

Proof. Let f be in $W^{(1)}(\Omega)$ and $\{\Omega_k\}_{k=1}^{\infty}$ be an exhaustion of Ω. By Remark 1, $\int f(y)d\varepsilon^{(i)}_{x,c\Omega_k}(y)=\int H^{\Omega_k}_f(z)G^{(i-1)}_{\Omega_k}(x,z)dz$. Since f is in $W^{(i)}(\Omega)$, $h^{(i)}_f(x)=\int h^{(1)}_f(z)G^{(i-1)}_{\Omega}(x,z)dz$. Hence $(-\Delta)^{i-1}h^{(i)}_f(x)=h^{(1)}_f(x)$ in Ω and $h^{(1)}_f$ being harmonic, we have $\Delta^i h^{(i)}_f=0$ in Ω.

Lemma 3. *If Ω satisfies the condition* [i] *and f is in $W^{(1)}(\Omega)$, then $f-(-\Delta)^{i-1}h^{(i)}_f$ is in $W^{(1)}_0(\Omega)$.*

Proof. By Lemma 2, $(-\Delta)^{i-1}h^{(i)}_f$ is harmonic, so it is in $W^{(1)}(\Omega)$ and $f-(-\Delta)^{i-1}h^{(i)}_f$ is in $W^{(1)}(\Omega)$.

Moreover for an exhaustion $\{\Omega_k\}_{k=1}^{\infty}$ of Ω we have

$$\begin{aligned}&\int(f(y)-(-\Delta)^{i-1}h_f^{(i)}(y))d\varepsilon_{x,c\Omega_k}^{(i)}(y)\\&=\int f(y)d\varepsilon_{x,c\Omega_k}^{(i)}(y)-\int H_{(-\Delta)^{i-1}h_f^{(i)}}^{\Omega_k}(z)G_{\Omega_k}^{(i-1)}(x,z)dz\\&=\int f(y)d\varepsilon_{x,c\Omega_k}^{(i)}(y)-\int(-\Delta)^{i-1}h_f^{(i)}(z)G_{\Omega_k}^{(i-1)}(x,z)dz,\end{aligned}$$

then

$$\begin{aligned}&\lim_{k\to\infty}\int(f(y)-(-\Delta)^{i-1}h_f^{(i)}(y))d\varepsilon_{x,c\Omega_k}^{(i)}(y)\\&=h_f^{(i)}(x)-\int(-\Delta)^{i-1}h_f^{(i)}(z)G_{\Omega}^{(i-1)}(x,z)dz=h_f^{(i)}(x)-h_f^{(i)}(x)=0,\end{aligned}$$

and so $f-(-\Delta)^{i-1}h_f^{(i)}$ is in $W_0^{(i)}(\Omega)=W_0^{(1)}(\Omega)$.

3. In this section we always assume that Ω is an open set satisfying the condition [m].

Let Ω_W^* be a Wiener compactification of Ω, $\Delta_W=\Omega_W^*-\Omega$ and Γ_W be a harmonic boundary of Ω_W^* (cf. [1]). We note that $\Gamma_W=\{x\in\Delta_W\,;\,f(x)=0$ for any f in $W_0^{(1)}(\Omega)\}$.

We shall show the following

Theorem 2. *Let $\{\Omega_k\}_{k=1}^{\infty}$ be an exhaustion of Ω, then the sequence $\{\varepsilon_{x,c\Omega_k}^{(i)}\}_{k=1}^{\infty}$ converges vaguely in Ω_W^* for each point x in $\Omega(1\leq i\leq m)$, and if we denote by $\varepsilon_x^{(i)}$ its limit, then*

$$S_{\varepsilon_x^{(i)}}=S_{\varepsilon_x^{(1)}}=\Gamma_W(2\leq i\leq m),$$

where $S_{\varepsilon_x^{(i)}}$ is the support of the measure $\varepsilon_x^{(i)}$.

Proof. We know that $W^{(1)}(\Omega)=C(\Omega_W^*)$ (cf. [1] Satz 9.3), so for any function in $C(\Omega_W^*)$ its restriction on Ω is in $W^{(i)}(\Omega)$.

Therefore the sequence $\left\{\int f(y)d\varepsilon_{x,c\Omega_k}^{(i)}(y)\right\}_{k=1}^{\infty}$ converges for any f in $C(\Omega_W^*)$. Next we shall show $S_{\varepsilon_x^{(i)}}\subset\Gamma_W$. Let f be a bounded continuous function on Δ_W such that $f=0$ on Γ_W and f^* be a continuous function on Ω_W^* such that $f^*=f$ on Δ_W. Then f^* being in $W_0^{(i)}(\Omega)$,

$$\int f(y)d\varepsilon_x^{(i)}(y)=\lim_{k\to\infty}\int f^*(y)d\varepsilon_{x,c\Omega_k}^{(i)}(y)=0.$$

This means $S_{\varepsilon_x^{(i)}}\subset\Gamma_W$. Since $S_{\varepsilon_x^{(1)}}=\Gamma_W$ (cf. [1] Satz 8.5), it suffices to show $S_{\varepsilon_x^{(1)}}\subset S_{\varepsilon_x^{(i)}}$.

Let f be in $C(\Delta_W)$ and f^* be in $C(\Omega_W^*)$ such that $f^*=f$ on Δ_W. Then for each k,

$$\int f^*(y)d\varepsilon_{x,c\Omega_k}^{(i)}(y)=\int\left(\int f^*(y)d\varepsilon_{z,c\Omega_k}^{(1)}(y)\right)G_{\Omega_k}^{(i-1)}(x,z)dz.$$

Since the restriction f^* on Ω is in $W^{(1)}(\Omega)$, we have

$$\int f(y)d\varepsilon_x^{(i)}(y)=\int\left(\int f(y)d\varepsilon_z^{(1)}(y)\right)G_{\Omega}^{(i-1)}(x,z)dz.$$

and so $S_{\varepsilon_x^{(1)}}\subset S_{\varepsilon_x^{(i)}}$.

Finally we shall treat Riquier's problem for an open set Ω satisfying the condition [m] and show the following

Theorem 3. *Let f_i be in $C(\Delta_W)$ $(1\leq i\leq m)$. Then there exists a function $h_{(f_1,f_2,\cdots,f_m)}$ in Ω such that*

$$\Delta^m h_{(f_1,f_2,\cdots,f_m)}=0$$

in Ω and for each $i(1\leq i\leq m)$, on Γ_W

$$(-\Delta)^{i-1}h_{(f_1,f_2,\cdots,f_m)}=f_i.$$

Proof. Let f_i^* be in $C(\Omega_W^*)$ such that $f_i^*=f_i$ on $\Delta_W(1\leq i\leq m)$. Then by Corollary 1 the restriction of f_i^* on Ω is in $W^{(i)}(\Omega)$ and so

$$h_{f_i^*}^{(i)}(x)=\int f_i(y)d\varepsilon_x^{(i)}(y).$$

We put

$$h_{(f_1,f_2,\cdots,f_m)}(x)=\sum_{i=1}^m h_{f_i^*}^{(i)}(x),$$

then by Lemma 2, $h_{(f_1,f_2,\cdots,f_m)}$ is polyharmonic of order m in Ω and for each $i(1\leq i\leq m)$,

$$(-\Delta)^{i-1}h_{(f_1,f_2,\cdots,f_m)}(x)=(-\Delta)^{i-1}h_{f_i^*}^{(i)}(x)+\sum_{k=i+1}^m(-\Delta)^{i-1}h_{f_k^*}^{(k)}(x).$$

Since

$$\sum_{k=i+1}^m(-\Delta)^{i-1}h_{f_k^*}^{(k)}(x)=(-\Delta)^{i-1}\left(\sum_{k=i+1}^m\int h_{f_k^*}^{(1)}(y)G_\Omega^{(k-1)}(x,y)dy\right)$$
$$=\int\left(\sum_{k=1}^{m-i}(-1)^k\int h_{f_{i+k}^*}^{(1)}(z)G_\Omega^{(k-1)}(y,z)dz\right)G_\Omega(x,y)dy,$$

we know it is in $W_0^{(1)}(\Omega)$. On the other hand $(-\Delta)^{i-1}h_{f_i^*}^{(i)}$ is in $W^{(1)}(\Omega)$, hence $(-\Delta)^{i-1}h_{(f_1,f_2,\cdots,f_m)}$ can be continuously extended over Ω_W^*.

For each point x in Γ_W, we have

$$(-\Delta)^{i-1}h_{(f_1,f_2,\cdots,f_m)}(x)=(-\Delta)^{i-1}h_{f_i^*}^{(i)}(x)$$
$$=h_{f_i^*}^{(1)}(x)=f_i(x).$$

Remark 2. Conversely if Ω has such a solution as $h_{(f_1,f_2,\cdots,f_m)}$, then Ω satisfies the condition [m].

Remark 3. If Ω is an unbounded open subset of $R^n(n\geq 3)$ with finite volume, then $\sup_{x\in\Omega}\int G_\Omega(x,y)dy<+\infty$ and for any point x in Ω

$$\int G_\Omega^{(m-1)}(x,y)dy\leq\left(\sup_{x\in\Omega}\int G_\Omega(x,y)dy\right)^{m-1}<+\infty.$$

Hence Ω satisfies the condition [m]. Moreover the point at infinity being contained in $S_{\varepsilon_x^{(i)}}$ for each $i(1\leq i\leq m)$, the above theorem may be an extension of Lemma 1.

4. Let Ω be an open subset of $R^n(n\geq 2)$ and p be a non-negative continuously differentiable function on Ω.

We consider on Ω an elliptic differential equation:

$$L_p u=(\Delta-p)u=\Delta u-pu=0.$$

We shall see that in this case, similarly to the equation $\Delta u=0$, the above discussions succeed.

If we assume that p is bounded from below by a positive constant, for the Green function $G_{p,\Omega}$ of the equation $L_p u=0$ on Ω, we have

$$\sup_{x\in\Omega}\int G_{p,\Omega}(x,y)dy<+\infty.$$

Hence in this case we know

$$\int G_{p,\Omega}^{(m-1)}(x,y)dy\leq\left(\sup_{x\in\Omega}\int G_{p,\Omega}(x,y)dy\right)^{m-1}<+\infty$$

for each point x in Ω, where

$$G_{p,\Omega}^{(m-1)}(x,y)=\int\cdots\int G_{p,\Omega}(x,z_1)G_{p,\Omega}(z_1,z_2)\cdots G_{p,\Omega}(z_{m-2},y)dz_1dz_2\cdots dz_{m-2},$$

and so the condition [m] is satisfied in this case.

Let $\Omega^*_{W_p}$ be a p-Wiener compactification of Ω, Δ_{W_p} be its ideal boundary and Γ_{W_p} be its harmonic boundary (cf. [3]), then we shall have analogously to Theorem 3 the following

Theorem 4. *We assume that p is bounded from below by a positive constant. Let f_i be in $C(\Delta_{W_p})$ $(1\leq i\leq m)$. Then there exists a function $h^p_{(f_1,f_2,\cdots,f_m)}$ on Ω such that $(L_p)^m h^p_{(f_1,f_2,\cdots,f_m)}=0$ in Ω and for each $i(1\leq i\leq m)$, on Γ_{W_p}*

$$(-L_p)^{i-1}h^p_{(f_1,f_2,\cdots,f_m)}=f_i.$$

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