244. *A Criterion for Boundedness of a Linear Map from any Banach Space into a Banach Function Space**)

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(Comm. by Kinjirô KUNUGI, M. J. A., Dec. 12, 1970)

[**Abstract.** Let (X, Λ, μ) be a totally sigma-finite complete measure space, ρ be a function norm with associate seminorm ρ', Y be a Banach space. If $L\rho$ is a Banach function space then for a linear mapping $T: Y \to L\rho$ be continuous it is necessary and sufficient that given $E \in \Lambda$ with $\rho'(\chi_E) < \infty$ the functional T_E defined by $T_E y = \int (Ty)(x)\chi_E(x)d\mu(x)$ is continuous. It is noted that the collection $\{E \in \Lambda: \mu(E), \rho'(\chi_D) < \infty\}$ is sufficient to generate the same integration theory as Λ and if ρ satisfies the Fatou property this collection even generates (algebraically and isometrically) the function space $L\rho$.]

This note is based entirely on the notes of Luxemburg and Zaanen [13], a knowledge of which, will be assumed throughout; the notations of those authors will be preserved and references to [13] Will simply note the particular results of [13] without further modification. Of course, reference to papers other than [13] will be modified by the appropriate reference list number.

Theorem. *Let ρ be a function norm satisfying the Riesz-Fischer property (so $L\rho$ is a Banach function space); suppose that Y is a Banach space and that $T: Y \to L\rho$ is a linear mapping.*

Then in order that T be continuous it is necessary and sufficient that the following hold: given $E \in \Lambda$ such that $\chi_E \in L\rho'$, the linear functional T_E defined on Y to the scalar field by

$$T_E y = \int_E (Ty)(x)d\mu(x)$$

be a mumber of Y'.

Proof. Necessity follows immediately from Lemma 13.1.

To prove sufficiency, we note that since ρ is a function norm it follows from Corollary 11.5 that ρ' is saturated (in fact, ρ's being a norm is equivalent to ρ''s being saturated), so that by Theorem 8.7 there exists a sequence of subsets X_n of X satisfying $X_n \nearrow X$, $\mu(X_n) < \infty$, and $\rho'(\chi_{X_n}) < \infty$ (of course, $X_n \in \Lambda$; for the rest of the proof we will assume the sequence $\{X_n\}$ to be chosen according to these requirements.

We now consider the linear mapping $T: Y \to L\rho$. We will show

*) The research for this paper was supported in part by West Georgia College Faculty Research Grant No. 699.

that T is continuous by showing its graph to be closed, whence as Y and $L\rho$ are Banach spaces we can apply the closed graph theorem to yield T's continuity.

Let $y_m \in Y$ for $m=0,1,\cdots$ and $f_m \in L\rho$ for $m=0,1,\cdots$. Suppose for $m=1,2,\cdots$, that $Ty_m=f_m$ and that $y_m \to y_0$ (in Y) and $f_m \to f_0$ (in $L\rho$-norm). Fix n momentarily and consider the set X_n. For each set $E \in \Lambda$, such that $E \subseteq X_n$, we have $\rho'(\chi_E) \leq \rho'(\chi_{X_n}) < \infty$, so that the linear functional $T_E \in Y'$ in particular,

$$\begin{aligned}\int_E (Ty_0)(x)d\mu(x) &= T_E y_0 \\ &= \lim_m T_E y_m \\ &= \lim_m \int_E (Ty_m)(x)d\mu(x) \\ &= \lim_m \int_E f_m(x)d\mu(x) \\ &= \int_E f_0(x)d\mu(x)\end{aligned}$$

by the fact that (by Lemma 13.1) convergence in $L\rho$ is stronger than the L_1-convergence on E's for which $\rho'(\chi_E) < \infty$. Thus, for each $E \in \Lambda$, satisfying $E \subseteq X_n$ we have

$$\int_E (Ty_0)(x)d\mu(x) = \int_E f_0(x)d\mu(x).$$

Thus, by the Radon-Nikodym theorem $Ty_0 = f_0$(μ-a.e.) on X_n.

But then,

$$\begin{aligned}Ty_0 &= \lim_n \chi_{X_n} \cdot Ty_0 \\ &= \lim_n \cdot \chi_{X_n} \cdot f_0 = f_0\end{aligned}$$

holds μ-a.e. Thus, Yy_0 and f_0 are—as members of $L\rho$-identical and T's graph is closed.

Several remarks seem appropriate; they are in a sense directly related to the above theorem while they might be considered to be of some independent interest.

Suppose ρ is any function norm. Define $V\rho$ to be the collection

$$\{E \in \Lambda : \mu(E), \rho'(\chi_E) < \infty\}$$

It is clear that $V\rho$ is a sub-ring of Λ (in fact, $V\rho$ is even an ideal in the Boolean algebra Λ [12]) which contains all the μ-null sets and (as is readily seen using Theorem 6. B of [12], along with Corollary 11.5 and Theorem 8.7) generates all of Λ.

Suppose we denote by $v\rho$ the restriction of μ to $V\rho$. Then, in the terminology of Bogdanowicz (1), triple $(X, V\rho, v\rho)$ forms a volume space. In a sequence of papers ([1], [2], [3], [4], and [5]), Bogdanowicz has developed an approach to the theory of integration and measurable functions generated by a volume space; in another sequence of papers ([6], [7], [8], and [9]), he related the above approach to the Classical measure-theoretic approaches and gave necessary and sufficient conditions for different volumes and measures to generate the same (algebraically and isometrically) classes of Lebesgue-Bochner summable functions and identical (generalized) Lebesgue-Bochner-Stieltjes type integrals.

Using his results, as well as, the above remarks on the volume space $(X, V\rho, v\rho)$, it is a painless exercise to establish that the spaces of Lebesgue-Bochner integrable functions generated by the volume space $(X, V\rho, v\rho)$ and the measure space (x, Λ, μ) are the same (algebraically and isometrically).

It follows from this, using the techniques of the papers cited above, that the spaces of Lebesgue-Bochner measurable functions, L^p-spaces and even the Orlicz spaces of Lebesgue-Bochner measurable functions are identical (algebraically and, when applicable, isometrically) whether generated by $(X, V\rho, v\rho)$ or (X, Λ, μ).

Pursueing this train of thought a bit further, note that if ρ also satisfies the Fatou property then ρ is definable by means of integrals of members of its associate space $L\rho'$; in fact, $\rho=\rho''$ so $\rho(f) = \sup\left\{\int |f|\, g d\mu : \rho'(g) \leq 1\right\}$ note that ρ' satisfies the Fatou property, thus by Theorem 20.B of [12], we may assume the g's in the above definition to be simple functions; again, it is easily shown that $(X, V\rho, v\rho)$ generates the same space $L\rho$ as does (X, Λ, μ) in the sense that given $f \in L\rho$ (X, Λ, μ) then $\rho(f)$ can be written in the form

$$\rho(f) = \sup\left\{\int |f|\, s dv\rho\right\}$$

where the supremum is taken over all $V\rho$-simple functions s such that $\rho'(s) \leq 1$, and the above integral is the one discussed in [3].

Finally we remark that the above theorem is in a certain sense an improvement of the results contained in Gretsky's Memoir [11] on pages 11–19; it is proved under considerably more general hypotheses on the function norm ρ than Grestsky's theorems. However, Gretsky's results are considerably more pleasing in the sense that he (through use of the additional hypotheses) obtains precise estimates involving the norm of the operator T as an operator and the norm of a related set function defined on $V\rho$ (which involves only the constant found in Amemiya's theorem (5.5)); of course, this is done by avoiding use of the closed graph theorem.

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