28. *Angular Cluster Sets and Horocyclic Angular Cluster Sets*

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1. In [1] Bagemihl began a study of relations between non-tangential (angular) boundary behaviors and horocyclic boundary behaviors of meromorphic functions defined in the open unit disk D of the complex plane. This study has been continued by Dragosh in [2] and [3]. The purpose of the present paper is to sharpen some of results of these investigations by the method of Dolzhenko's paper.

Notation and definitions. Unless otherwise stated, $f: D \to W$ shall mean $f(z)$ is an arbitrary function (generally not unique) defined in the open unit disk $D: |z|<1$ and assuming values in the extended complex plane W. The unit circle $|z|=1$ is denoted by Γ.

A circle internally tangent to Γ at a point $\zeta \in \Gamma$ is called a horocycle at ζ, and will be denoted by $h_r(\zeta)$, where r $(0<r<1)$ is the radius of the horocycle.

Given a horocycle $h_r(\zeta)$ at a point $\zeta \in \Gamma$, the region interior to $h_r(\zeta)$ is called an oricycle at ζ, and will be denoted by $K_r(\zeta)$, or simply $K(\zeta)$ without specifying r. The half of $K_r(\zeta)$ lying to the right of the radius at ζ as viewed from the origin will be denoted by $K_r^+(\zeta)$ and $K_r^-(\zeta)$ denotes the left half of $K_r(\zeta)$ analogously.

Suppose that $0<r_1<r_2<1$. Let $r_3 (0<r_3<1)$ be so large that the circle $|z|=r_3$ intersects both of the horocycles $h_{r_1}(\zeta)$ and $h_{r_2}(\zeta)$. We define the right horocyclic angle $H^+_{r_1,r_2,r_3}(\zeta)$ at ζ with radii r_1, r_2, r_3 to be

$$H^+_{r_1,r_2,r_3}(\zeta)=\mathrm{com}\,(\overline{K^+_{r_1}(\zeta)}) \cap K^+_{r_2}(\zeta) \cap \{z: |z| \geqq r_3\},$$

where the bar denotes closure and comp denotes complement, both relative to the plane. The corresponding left horocyclic angle is denoted $H^-_{r_1,r_2,r_3}(\zeta)$. We write $H_{r_1,r_2,r_3}(\zeta)$ to denote a hyrocyclic angle at ζ without specifying whether it be right or left, or simply $H(\zeta)$ in the event r_1, r_2, r_3 are arbitrary.

We assume the reader to be familiar with the rudiments of the cluster sets.

$C_V(f,\zeta)$, the angular cluster set of $f(z)$ at ζ on a Stolz angle $V(\zeta)$;

$C_K(f,\zeta)$, the oricyclic cluster set of $f(z)$ at ζ on an oricycle $K(\zeta)$;

$C_H(f,\zeta)$, the horocyclic angular cluster set of $f(z)$ at ζ on a horocyclic angle $H(\zeta)$.

A point $\zeta \in \Gamma$ is said to be a horocyclic angular Plessner point (oricyclic Plessner point) of $f(z)$ provided that

$$C_H^+(f,\zeta)=W \text{ and } C_H^-(f,\zeta)=W \ (C_K(f,\zeta)=W)$$

for each right and left horocyclic angle (each oricycle) at ζ.

A point $\zeta\in\Gamma$ is called a horocyclic angular Fatou point (oricyclic Fatou point) of $f(z)$ with a horocyclic angular Fatou value (an oricyclic Fatou value) $w\in W$ provided that

$$C_H^+(f,\zeta)=\{w\} \text{ and } C_H^-(f,\zeta)=\{w\} \ (C_K(f,\zeta)=\{w\})$$

for each right and left horocyclic angle (each oricycle) at ζ.

Suppose a set $A\subset\Gamma$ and a point $\zeta=e^{i\theta}\in\Gamma$ are given. For a $\varepsilon>0$, we denote an arc $\{e^{i\theta'}; \theta-\varepsilon<\theta'<\theta+\varepsilon\}$ by $\Gamma(\varepsilon,\zeta)$. Let $\gamma(\zeta,\varepsilon,A)$ be the largest of arcs contained in $\Gamma(\varepsilon,\zeta)$ and not intersecting with A. The set A is of porosity of the order α, $0<\alpha\leqq 1$ (or simply of porosity (α)) at ζ, if

$$\varlimsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\{\gamma(\zeta,\varepsilon,A)\}^{\alpha}>0.$$

A is of porosity (α) on Γ if it is so at each $\zeta\in A$. A set which is a countable sum of sets of porosity (α) is said to be of σ-porosity (α).

A set of σ-porosity (α) is of the first Baire category.

It is easily seen that a measurable set which is of porosity (1) on Γ has no points of density. Therefore every measurable set of σ-porosity (1) on Γ is of measure 0. But there exists the set, which is of measure 0 and not of σ-porosity (1) (see [6], p. 75).

2. A $KH(KV)$-singular point is the point $\zeta\in\Gamma$ such that $C_K(f,\zeta)\neq C_H(f,\zeta)(C_K(f,\zeta)\neq C_V(f,\zeta))$ for some pair of $K(\zeta)$ and $H(\zeta)(K(\zeta)$ and $V(\zeta))$. The set of all $KH(KV)$-singular points is denoted by $E_{KH}(f)$ $(E_{KV}(f))$.

A KK-singular point is the point $\zeta\in\Gamma$ such that $C_{K'}(f,\zeta)\neq C_{K''}(f,\zeta)$ for some pair of oricycles $K'(\zeta)$ and $K''(\zeta)$. The set of all KK-singular points is denoted by $E_{KK}(f)$.

Let $\{r_i\}_{i=1}^{\infty}$ be a sequence of all rational numbers satisfying $0<r_i<1$, and let $\{D_n\}$ be a sequence consisting of all closed circles of the plane W having rational radii r_n and centers with rational coordinates.

For a $\varepsilon>0$, we set $U_\varepsilon(\zeta)=\{z; |z-\zeta|<\varepsilon\}$. We denote $K_{r_p}(\zeta)$ by $K_p(\zeta)$ and $H_{r_k,r_l,r_m}(\zeta)$ by $H_{k,l,m}(\zeta)$.

Lemma 1. *Let $\zeta\in A\subset\Gamma$. Suppose A is not of porosity* (1) *at a point $\zeta\in A$. Then for fixed $r_p, r_k, r_l, r_m, K_p(\zeta)\cap U_\varepsilon(\zeta)$ is covered by the set $M=\bigcup_{\xi\in A}H_{k,l,m}(\xi)$ supposed $\varepsilon>0$ is sufficiently small.*

Proof. Without loss of generality, we may assume that $\zeta=1$. Now we suppose that there exists a sequence $z_\nu=x_\nu+iy_\nu(\nu=1,2,3,\cdots)$ such that $z_\nu\in K_p(1)\cap U_\varepsilon(1)-M$ and $z_\nu\to 1$. For each z_ν, points $R_1(z_\nu)$, $S_1(z_\nu)$, $R_2(z_\nu)$, $S_2(z_\nu)$ on Γ are decided as follows.

$R_1(z_\nu)(S_1(z_\nu))$ is the point on Γ such that the point z_ν lies on the right half of $h_{r_l}(R_1(z_\nu))(h_{r_k}(S_1(z_\nu)))$;

$R_2(z_\nu)(S_2(z_\nu))$ is the point on Γ such that the point z_ν lies on the left half of $h_{r_l}(R_2(z_\nu))(h_{r_k}(S_2(z_\nu)))$.

Let $z_\nu = r_\nu e^{i\theta_\nu}$. We immediately have

$\overline{R_1(z_\nu)S_1(z_\nu)}$ (the arc length connecting $R_1(z_\nu)$ and $S_1(z_\nu)$)

$$=\overline{R_2(z_\nu)S_2(z_\nu)}=\cos^{-1}\left\{\frac{2(1-r_l)-(1-r_\nu^2)}{2(1-r_l)r_\nu}\right\}-\cos^{-1}\left\{\frac{2(1-r_k)-(1-r_\nu^2)}{2(1-r_k)r_\nu}\right\},$$

$$\overline{R_i(z_\nu)\,1}=\theta_\nu-(-1)^i\cos^{-1}\left\{\frac{2(1-r_l)-(1-r_\nu^2)}{2(1-r_l)r_\nu}\right\}\qquad(i=1,2),$$

$$\overline{S_i(z_\nu)\,1}=\theta_\nu-(-1)^i\cos^{-1}\left\{\frac{2(1-r_k)-(1-r_\nu^2)}{2(1-r_k)r_\nu}\right\}\qquad(i=1,2).$$

Since $z_\nu\in K_p(1)$, $|\theta_\nu|<\cos^{-1}\left\{\frac{2(1-r_p)-(1-r_\nu^2)}{2(1-r_p)r_\nu}\right\}$. We set

$$\varepsilon_\nu=\max\{\overline{R_1(z_\nu)1},\overline{S_1(z_\nu)1},\overline{R_2(z_\nu)1},\overline{S_2(z_\nu)1}\}.$$

$$\lim_{\nu\to\infty}\frac{\overline{R_1(z_\nu)S_1(z_\nu)}}{\sqrt{1-r_\nu}}>0 \text{ and } \varepsilon_\nu=O(\sqrt{1-r_\nu}) \text{ as } \nu\to\infty.$$

Since $\{R_1(z_\nu)S_1(z_\nu)$ (the arc connecting $R_1(z_\nu)$ and $S_1(z_\nu)) \cup R_2(z_\nu)S_2(z_\nu)\}$ $\cap A=\phi$, we have $\gamma(1,\varepsilon_\nu,A)\geqq\overline{R_1(z_\nu)S_1(z_\nu)}$.

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\gamma(1,\varepsilon,A)\geqq\lim_{\nu\to\infty}\frac{1}{\varepsilon_\nu}\overline{R_1(z_\nu)S_1(z_\nu)}\geqq\lim_{\nu\to\infty}\frac{\sqrt{1-r_\nu}}{\varepsilon_\nu}\frac{\overline{R_1(z_\nu)S_1(z_\nu)}}{\sqrt{1-r_\nu}}>0,$$

and obtain a contradiction to the assumption that 1 is not a point of porosity (1) for A. Therefore, for $\varepsilon>0$ small enough, $K_p(1)\cap U_\varepsilon(1)$ is covered by the set $M=\bigcup_{\xi\in A}H_{k,l,m}(\xi)$.

Lemma 2 (Yanagihara [5, Theorem 1]). *Let $f:D\to W$. Then $E_{KK}(f)$ is of the type $G_{\delta\sigma}$ and σ-porosity* (1).

Theorem 1. *Let $f:D\to W$. Then $E_{KH}(f)$ is of the type $G_{\delta\sigma}$ and σ-porosity* (1).

Proof. $E_{n,k,l,m}$ is the set of points $\zeta\in\Gamma$ such that the set

$$\{w=f(z)\,;\,z\in H_{k,l,m}(\zeta)\}\tag{1}$$

lies at a distance $\geqq r_n$ from D_n.

$F_{n,p,q}$ is the set of points $\zeta\in\Gamma$ such that the set

$$\left\{w=f(z)\,;\,z\in K_p(\zeta),\frac{1}{3q}<\text{dis}\,(z,\zeta)<\frac{1}{q}\right\}\tag{2}$$

has common points with D_n.

Then $E_{n,k,l,m}$ is closed and $F_{n,p,q}$ is open. We put

$$F_{n,p}=\bigcap_{t=1}^{\infty}\bigcup_{q=t}^{\infty}F_{n,p,q} \text{ and } A_{n,k,l,m,}\cap F_{n,p}.\tag{3}$$

We will show

$$E_{KH}(f)=(\bigcup_{n,k,l,m,p}A_{n,k,l,m,p})\cup E_{KK}(f).\tag{4}$$

Take a point $\zeta\in E_{KH}(f)$ and $\zeta\notin E_{KK}(f)$. There exist $K(\zeta)$ and $H(\zeta)$, $K(\zeta)\supset H(\zeta)$, for which $C_K(f,\zeta)\supsetneqq C_H(f,\zeta)$. Choose number p and s such that $K_p(\zeta)\supset K(\zeta)$ and

$$D_s\cap C_K(f,\zeta)\neq\phi,\quad \text{dis}\,(D_s,C_H(f,\zeta))>5r_s.\tag{5}$$

Then we can find numbers k, l, m such that $H(\zeta) \supset H_{k,l,m}(\zeta)$ and $\operatorname{dis}(D_s, f(z)) > 4r_s$ for $z \in H_{k,l,m}(\zeta)$. If D_n is a disk with radius $r_n = 2r_s$ and concentric with D_s, $\operatorname{dis}(D_n, f(z)) > r_n$ for $z \in H_{k,l,m}(\zeta)$, which shows $\zeta \in E_{n,k,l,m}$. In view of (5) there exists an infinite number of q such that

$$D_n \cap \left\{ w = f(z) \, ; \, z \in K_p(\zeta), \frac{1}{3q} < \operatorname{dis}(z, \zeta) < \frac{1}{q} \right\} \neq \phi,$$

which shows $\zeta \in F_{n,p}$. Thus $\zeta \in A_{n,k,l,m,p}$ and

$$E_{KH}(\zeta) \subset \Big(\bigcup_{n,k,l,m,p} A_{n,k,l,m,p} \Big) \cup E_{KK}(f).$$

Take a point $\zeta \in \big(\bigcup_{n,k,l,m,p} A_{n,k,l,m,p} \big) \cup E_{KK}(f)$. If $\zeta \in E_{KK}(f)$, clearly we have $\zeta \in E_{KH}(f)$. If $\zeta \in \bigcup_{n,k,l,m,p} A_{n,k,l,m,p}$, $C_{H_{k,l,m}}(f, \zeta) \cap D_n = \phi$ from (1) and $C_{K_p}(f, \zeta) \cap D_n \neq \phi$ from (2). Thus we have $C_{H_{k,l,m}}(f, \zeta) \neq C_{K_p}(f, \zeta)$ and $\zeta \in E_{KH}(f)$. Hence $\big(\bigcup_{n,k,l,m,p} A_{n,k,l,m,p} \big) \cup E_{KK}(f) \subset E_{KH}(f)$.

Since $E_{KK}(f)$ is of type $G_{\delta\sigma}$ by Lemma 2, the equality (4) shows that $E_{KH}(f)$ is type $G_{\delta\sigma}$.

According to Lemma 1, $E_{KK}(f)$ is of σ-porosity (1), so that it remains to prove that $A = A_{n,k,l,m,p}$ is of porosity (1).

Suppose A is not of porosity (1) at a point $\zeta \in A$. Then for sufficiently small $\varepsilon > 0$, $K_p(\zeta) \cap U_\varepsilon(\zeta)$ is covered by the set $\bigcup_{\xi \in A} H_{k,l,m}(\xi)$ by Lemma 1. Thus if $z \in K_p(\zeta) \cap U_\varepsilon(\zeta)$, there is a point $\xi \in A = A_{n,k,l,m,p}$, $z \in H_{k,l,m}(\xi)$. Therefore $w = f(z)$ lies at a distance $\geqq r_n$ from D_n, and $C_{K_p}(f, \zeta) \cap D_n = \phi$. This contradicts with $\zeta \in F_{n,p}$. Thus the porosity (1) of A is proved.

Theorem 2 (Yanagihara [5, Theorem 2]). *Let $f : D \to W$. Then $E_{KV}(f)$ is of the type $G_{\delta\sigma}$ and of σ-porosity* (1/2).

3. Now we can state some precisions and generalizations of the results of Bagemihl [1, Theorem 1, Theorem 2, Theorem 4 and Remark 3] and Dragosh [3, Theorem 1, Remark 2 and Corollary 2].

Theorem 3. *Let $f : D \to W$. Then a horocyclic angular Fatou point of $f(z)$ is an angular Fatou point of $f(z)$ except on a set of σ-porosity* (1).

Proof. According to Theorem 1, except on a set of σ-porosity (1), a horocyclic angular Fatou point of f is an oricyclic Fatou point of f, which is an angular Fatou point of f by the fact $C_K(f, \zeta) \supset C_V(f, \zeta)$.

Theorem 4. *Let $f : D \to W$. Then an angular Fatou point of $f(z)$ is a horocyclic angular Fatou point except on a set of σ-porosity* (1/2).

Proof. This is an analogous deduction from Theorem 2.

Theorem 5. *Let $f : D \to W$. Then an angular Plessner point of $f(z)$ is a horocyclic angular Plessner point of $f(z)$ except on a set of σ-porosity* (1).

Proof. By the fact $C_K(f, \zeta) \supset C_V(f, \zeta)$, an angular Plessner point of f is an oricyclic Plessner point of f, which is a horocyclic angular Plessner point of f except on a set of σ-porosity (1) according to Theorem 1.

Theorem 6. *Let $f: D \to W$. Then a horocyclic angular Plessner point of $f(z)$ is an angular Plessner point of $f(z)$ except on a set of σ-porosity* (1/2).

Proof. This is an analogous deduction from Theorem 2.

4. We have not yet established the complete structual characterizations of set $E_i (i=1,2,3,4)$ such that

$$\begin{cases} \text{a horocyclic angular Fatou point is an angular Fatou point on } E_1, \\ \text{an angular Fatou point is a horocyclic angular Fatou point on } E_2, \\ \text{an angular Plessner point is a horocyclic angular Plessner point on } E_3. \\ \text{a horocyclic angular Plessner point is an angular Plessner point on } E_4, \end{cases}$$

for some functions.

But we establish only one special result in this direction here

In proving next Theorem 7, we use the function $f(z)$ constructed by Yanagihara [5].

Lemma 3. *Let $E \subset \Gamma$ be a closed everywhere disconnected set. Then there exists a bounded holomorphic function $f(z)$ with the following properties:*

1) *At every $\zeta \in \mathrm{comp}(F)$, $f(z)$ is continuous. Therefore, ζ is both an angular Fatou point and a horocyclic angular Fatou point.*

2) *Each point $\zeta \in F$ is an angular Fatou point having an angular limit of modulus* 1.

3) *Each point $\zeta \in F$ at which the set F is of porosity* (1/2) *is not a horocyclic angular Fatou point.*

Proof. Comp (F) consists of a countable number of arcs $(\zeta'_\nu, \zeta''_\nu)$. For a constant $r (0<r<1)$, $h_r(\zeta'_\nu) \cap h_r(\zeta''_\nu) \neq \phi$ except at most finite number of ν's. Let $z'_{\nu,1} = z''_{\nu,1}$ be the one of intersection points of $h_r(\zeta'_\nu)$ and $h_r(\zeta''_\nu)$ which is nearer to Γ. For each exceptional index ν, let $z'_{\nu,1}$ be the left one of intersection points of $h_r(\zeta'_\nu)$ and $|z|=1-r$, and let $z''_{\nu,1}$ be the right one of intersection points of $h_r(\zeta''_\nu)$ and $|z|=1-r$. Next, let $z'_{\nu,n}$ be the point on $h_r(\zeta'_\nu)$ such that

$$\frac{1-|z'_{\nu,n}|}{|\zeta'_\nu - z'_{\nu,n}|} = \frac{1}{2} \frac{1-|z'_{\nu,n-1}|}{|\zeta'_\nu - z'_{\nu,n-1}|} \qquad (n=2,3,4,\cdots).$$

The sequence $\{z''_{\nu,n}\}$ on $h_r(\zeta''_\nu)$ is defined analogously.

Then the Blaschke product

$$f(z) = \prod \frac{\overline{z'_{\nu,n}}}{|z'_{\nu,n}|} \frac{z - z'_{\nu,n}}{1-\overline{z'_{\nu,n}} z} \prod \frac{\overline{z''_{\nu,n}}}{|z''_{\nu,n}|} \frac{z - z''_{\nu,n}}{1-\overline{z''_{\nu,n}} z}$$

has the properties asserted in Lemma 3.

Now, we shall prove here the property 3) only.

If we choose appropriate constants r_1, r_2, r_3 and $\zeta \in F$ is a point at which the set F is of porosity (1/2), $H_{r_1,r_2,r_3}(\zeta)$ contains an infinite number of points from $\{z'_{\nu,n}, z''_{\nu,n}\}$. Hence, for each point ζ at which the set F is of porosity (1/2), if ζ is a horocyclic Fatou point of $f(z)$, then the horocyclic Fatou value at ζ must be 0, so that by the theorem of Lindelöf, the angular Fatou value at ζ must be 0, too. But this contradicts the property 2).

Theorem 7. *For each set of σ-porosity* (1/2) $E\subset\Gamma$, *there exists a bounded analytic function $f(z)$ for which each $\zeta\in E$ is an angular Fatou point and is not a horocyclic angular Fatou point.*

Proof. E can be represented in the form of a countable sum of sets E_n nowhere dense on $\Gamma: E=\cup E_n$. Suppose that a set E' is a countable sum of closed everywhere disconnected sets $\overline{E_n}: E'=\cup\overline{E_n}$. Then E' can also be represented in the form of a not more than countable sum of closed sets F_k without common points (Dolzehnko [4], English translation, p. 8).

From this construction, it is evident that each point $\zeta\in\overline{E_n}$ at which some E_n is of porosity (1/2) is also a point of porosity (1/2) for some F_k.

Now, for each F_k we construct a sequence of zeros $\{z^k_{\nu,n}\}$ and Blsschke product $f_k(z)$, as in Lemma 3. Set

$$f(z)=\sum 2^{-k}f_k(z).$$

If $\zeta\in E_k$, all $f_\nu(z)(\nu\neq k)$ are continuous at ζ (Lemma 3, 1)), and $f_k(z)$ has an angular limit of modulus 1 (Lemma 3, 2)). Hence each point $\zeta\in E$ is an angular Fatou point. On the other hand, if $\zeta\in E_n$ is a point of porosity (1/2) for F_k, then ζ is not a horocyclic angular Fatou point of $f_k(z)$ (Lemma 3, 3)). Thus each point $\zeta\in E$ is not a horocyclic angular Fatou point of $f(z)$.

Theorem 7 corresponds to Theorem 4. In this connection, it is natural to ask the following question: Is it true that there fold analogous results to Theorem 7 corresponding to other theorems in section 3? I would guess that it is positive even for holomorphic functions.

References

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