179. On Countably R-closed Spaces. II

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A topological space S is called *countably R-closed*, if for any family $\{G_n\}_{n=1}^{\infty}$ of nonvoid open sets such that $G_n \supset \bar{G}_{n+1}$ for every n, we have $\bigcap_{n=1}^{\infty} G_n \neq \phi$. Z. Frolik [1] and the present author [2] gave characterizations of countably R-closed spaces.

In any topological space S, if a family Φ, composed of subsets of S, has a point x such that each neighbourhood of x meets infinitely many members of Φ, we say that Φ *cluster to* x and that the point x is *a cluster point of* Φ. S. Kasahara [3] proved the following:

Proposition. *In any regular T_1-space S, the following conditions are equivalent:*

(i) *Every family of pairwise disjoint open sets has at least one cluster point.*

(ii) *Every star-finite open covering of S has a finite subcovering.*

(iii) *Every star-finite open covering of S has finite subfamily whose union is dense in S.*

We shall give another characterization of countably R-closed regular spaces, using the method of S. Kasahara [3].

In any topological space S, a family Φ composed of subsets of S is called *locally finite* if every point x has a neighbourhood $U(x)$ which meets only finite members of Φ, and Φ is called *star-finite* if every member of Φ meets only finite members of Φ. A subset E is called *regularly closed* if E is the closure of an open set of S. A covering of S composed of regularly closed sets is called *a regularly closed covering of* S.

Theorem. *In any regular space S, the following conditions are equivalent:*

(1) *S is countably R-closed.*

(2) *Every family of pairwise disjoint regularly closed sets has at least one cluster point.*

(3) *Every family of pairwise disjoint open sets has at least one cluster point.*

We shall prove that (1)→(2)→(1) and (2)→(3)→(2). In stead of (1), we shall use (4) and (5) of the following Lemma 1.

Lemma 1. *In any topological space S, the following conditions are equivalent:*

(1) *S is countably R-closed.*

(4) *Every locally finite, star-finite, countable, regularly closed*

covering of S has a finite subcovering.

(5) *Every locally finite, star finite, countable, regularly closed covering of S is a finite covering.*

(6) *Every locally finite, star-finite, regularly closed covering of S is a finite covering.*

(7) *Every star-finite open covering of S is a finite covering*

(Z. Frolik [1]).

Proof. This lemma is the theorem in [2].

Lemma 2. *In a topological space S, if there is a family $\{G_n\}_{n=1}^{\infty}$ of open sets where $G_n \supset \bar{G}_{n+1}$ for every n and $\bigcap_{n=1}^{\infty} G_n = \phi$, the family $\{\bar{H}_n\}_{n=0}^{\infty}$ where $H_0 = S - \bar{G}_1$ and $H_n = G_n - \bar{G}_{n+1}$ for every $n (\geqq 1)$, is locally finite, star-finite, regularly closed covering of S.*

Proof. This lemma is Lemma 3 in [2].

Proof that (5)→(2). Let S be a regular space and let $\{\bar{O}_\alpha\}_{\alpha \in A}$ be a infinite family of pairwise disjoint regularly closed sets. Let us assume that $\{\bar{O}_\alpha\}_{\alpha \in A}$ has no cluster point. Then the family $\{\bar{O}_\alpha\}_{\alpha \in A}$ is locally finite. Let $\{\bar{O}_n\}_{n=1}^{\infty}$ be a countably infinite subfamily of $\{\bar{O}_\alpha\}_{\alpha \in A}$. Since the space S is regular, there is a nonvoid open set V_2^2 such that $O_n \supset \bar{V}_2^2 \supset V_2^2$, that is $O_2 \supset V_2^2$, and nonvoid open sets V_n^m $m = 2, 3, \cdots, n$ such that $O_n \supset V_n^2 \supset V_n^3 \supset \cdots \supset V_n^m$ for every n $(\geqq 3)$. Furthermore, for every n $(\geqq 1)$ let us define $V_n^1 = O_n$, $V_n^m = \phi$ for every m $(\geqq n+1)$, and $G_m = \bigcup_{n=1}^{\infty} V_n^m$ for every m $(\geqq 1)$. By the locally finiteness of $\{\bar{O}_\alpha\}_{\alpha \in A}$, we have $\bar{G}_m = \bigcup_{n=1}^{\infty} \bar{V}_n^m$, hence $G_m \supset \bar{G}_{m+1} \supset G_{m+1}$ for every $m (\geqq 1)$. Since $\{\bar{O}_n\}_{n=1}^{\infty}$ has no cluster point, it is obvious that $\bigcap_{m=1}^{\infty} G_m = \phi$. Put $H_0 = S - \bar{G}_1$ and $H_m = G_m - \bar{G}_{m+1}$ for every $m \geqq 1$. In virtue of Lemma 2, the family $\{\bar{H}_m\}_{m=0}^{\infty}$ is a locally finite, star-finite, infinite covering of S, contrary to the property (5).

Proof that (2)→(4). Let S be a topological space and let $\{\bar{O}_n\}_{n=1}^{\infty}$ be a locally finite, star-finite, regularly closed covering of S. Let us assume that $\{\bar{O}_n\}_{n=1}^{\infty}$ has no finite subcovering, then $\bigcup_{n=1}^{m} \bar{O}_n \neq S$ for any finite m. From this, we assume that $\bar{O}_m - \bigcup_{n=1}^{m-1} \bar{O}_n \neq \phi$ for every $m \geqq 2$, without loss generality. Let us define $\bar{O}_{n_1} = \bar{O}_1$ and let us assume that $\bar{O}_{n_t}$, $t = 1, 2, \cdots, s$, are pairwise disjoint. According to the star-finiteness of $\{\bar{O}_n\}_{n=1}^{\infty}$, there is an $\bar{O}_m$ such that $m > n_s$ and $\bar{O}_m \cap \bigcup_{t=1}^{s} \bar{O}_{n_t} = \phi$. Put $\bar{O}_{n_{s+1}} = \bar{O}_m$. Thus we have obtained an infinite family $\{\bar{O}_{n_t}\}_{t=1}^{\infty}$ of pairwise disjoint regularly closed sets. Being locally finite, $\{\bar{O}_{n_t}\}_{t=1}^{\infty}$ has no cluster point, contrary to (2).

Proof that (2)→(3). The implications (2)⇄(3) are deduced by the proposition i.e. (3)⇄(7), the implications (5)→(2)→(4)→(5) and (5)⇄(6)⇄(7). We shall prove directly that (2)→(3). Let S be a regular space and let us assume that $\{O_\alpha\}_{\alpha \in A}$ is a family of pairwise disjoint open sets of S. For every O_α, there exists a regularly closed set $\bar{V}_\alpha$

such that $\bar{V}_\alpha \subset O_\alpha$, hence $\{\bar{V}_\alpha\}_{\alpha\in A}$ is a family of pairwise disjoint regularly closed sets. By the condition (2), $\{\bar{V}_\alpha\}_{\alpha\in A}$ has at least one cluster point which is a cluster point of $\{O_\alpha\}_{\alpha\in A}$.

Proof that (3)$\to$(2). We shall prove directly that (3)$\to$(2). Let S be a topological space and let us suppose that $\{\bar{O}_\alpha\}_{\alpha\in A}$ is a family of pairwise disjoint regularly closed sets. Then $\{O_\alpha\}_{\alpha\in A}$ is a family of pairwise disjoint open sets. By the condition (3), the family $\{O_\alpha\}_{\alpha\in A}$ has at least one cluster point which is a cluster point of $\{\bar{O}_\alpha\}_{\alpha\in A}$.

References

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