232. A Note on Spaces with a Uniform Base

By Takanori SHIRAKI
Ehime University

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In this note, we shall consider some properties in connection with the spaces with a uniform base. The notion of a uniform base was introduced by Aleksandrov [1]. A collection $\mathcal{B}$ of open sets in a space X is a *uniform base* if for each $x \in X$, any infinite subset of $\mathcal{B}$, each member of which contains x, is a local base at x. In [1] it is proved that a space X has a uniform base if and only if X has a development consisting of point-finite open coverings of X. Arhangel'skii [2] obtained that a T_1-space X has a uniform base if and only if X is an open, compact (continuous) image of some metric space. From these facts, it is known that a T_1-space X has a uniform base if and only if X is a metacompact (=point-paracompact), developable space. Also it is clear that a space with a uniform base has a σ-point-finite base. However, Example 6.4 of [8] shows that the converse of this result is not true in general (cf. [3]). Spaces are assumed to be T_1.

1. Characterizations of spaces with a uniform base. Recently the author has been informed that F. Siwiec has proved the following: *A T_1-space X has a uniform base if and only if X has a σ-point-finite base and each closed set of X is a G_δ-set,* and that he has asked to prove directly that the above condition for X implies X being an open, compact image of a metric space. We shall prove this in the proof of the following Theorem 1 which contains other characterizations of spaces with a uniform base.

Theorem 1. *For a T_1-space X, the following conditions are equivalent:*

1) *X is an open, compact image of a metric space,*
2) *X is a metacompact Σ-space with a point-countable base,*
3) *X is a $w\Delta$-space with a σ-point-finite base,*
4) *X is a $\Sigma^{\#}$-space with a σ-point-finite base.*

Proof. 1)→2). It is easy to show that X has a development $\{\mathcal{V}_i : i=1, 2, \cdots\}$ consisting of point-finite open coverings of X. Therefore X is metacompact and developable. Since X is a developable space, X has a σ-locally finite closed net and hence is a Σ-space.

2)→3). Since X is a T_1 Σ-space with a point-countable base, X is a developable space by [16, Corollary 1.3] and hence X is a $w\Delta$-space. Since X is a metacompact developable space, X has a σ-point-finite base.

3)$\to$4). Every T_1 space with a σ-point-finite base is an α-space (cf. [11]). Since X is a T_1 $w\Delta$-space and an α-space, X is semi-stratifiable and hence subparacompact. Then X is a Σ-space by [16, Corollary 2.2], therefore X is a $\Sigma^{\#}$-space.

4)$\to$1). Since X is a $\Sigma^{\#}$-space, X has a $\Sigma^{\#}$-net $\{\mathcal{F}_i\}$, that is, each $\mathcal{F}_i$ is a closure-preserving closed covering of X and every sequence $\{x_i\}$ such that $x_i \in C(x, \mathcal{F}_i)$ $(i=1, 2, \cdots)$ has a cluster point, where we denote $C(x, \mathcal{F}_i) = \cap \{F : x \in F \in \mathcal{F}_i\}$. Let $\bigcup_i \mathcal{B}_i$ be a σ-point-finite base such that each $\mathcal{B}_i$ is an open covering of X. For each $x \in X$ and each $n \in \{1, 2, \cdots\}$, we put $V_n(x) = X - \cup \{F : x \notin F \in \mathcal{F}_i, 1 \leq i \leq n\}$ and $B_n(x) = \cap \{B : x \in B \in \mathcal{B}_i, 1 \leq i \leq n\}$. Let $G_n(x) = B_n(x) \cap V_n(x)$. Then for each $x \in X$, we have the sequence $\{G_n(x)\}$ of open neighborhoods of x, satisfying i) $\bigcap_n G_n(x) = x$ and ii) if $x \in G_n(x_n)$ for each n, then the sequence $\{x_n\}$ converges to x. Therefore X is semi-stratifiable by Creede [9]. Since X is semi-stratifiable, each closed set of X is a G_δ. We shall prove directly that X is an open compact image of a metric space. For each n and each k, let $\mathcal{U}_{nk} = \left\{U : U = \bigcap_{i=1}^{k} B_i \neq \phi,\ B_i\text{'s are distinct elements of } \mathcal{B}_n\right\}$ and let $U_{nk} = \cup \{U : U \in \mathcal{U}_{nk}\}$. Since $X - U_{nk}$ is a G_δ-set, we have $X - U_{nk} = \bigcap_{j=1}^{\infty} V_{nkj}$, where V_{nkj} is an open set of X for each j. Let $\mathcal{U}_{nkj} = \mathcal{U}_{nk} \cup \{V_{nkj}\}$. Then it is a point-finite open covering of X. We order the collection $\{\mathcal{U}_{nkj} : n, k, j = 1, 2, \cdots\}$ in a sequence $\{\mathcal{L}_i : i = 1, 2, \cdots\}$ and denote $\mathcal{L}_i = \{U_\alpha : \alpha \in \Omega_i\}$ for each i. Then $\{\mathcal{L}_i\}$ has the following property: For each $x \in X$, if $x \in U_{\alpha_i} \in \mathcal{L}_i$ $(i = 1, 2, \cdots)$, then $\{U_{\alpha_i} : i = 1, 2, \cdots\}$ is a local base of x. Let $\prod_i \Omega_i$ be the product space, where Ω_i is endowed with the discrete topology, and let $A = \{a = (\alpha_i) \in \prod_i \Omega_i : \{U_{\alpha_i} : i = 1, 2, \cdots\}$ is a local base at some point of $X\}$. Then A is clearly a zero-dimensional metric space. We define a mapping f from A to X such that $f(a) = \bigcap_{i=1}^{\infty} U_{\alpha_i}$, where $a = (\alpha_i)$. It is not so hard to see that f is continuous, open and surjective. Since $f^{-1}(x) = \prod_i \{\alpha \in \Omega_i : x \in U_\alpha\}$ and $\{\alpha \in \Omega_i : x \in U_\alpha\}$ is finite, $f^{-1}(x)$ is compact. Hence f is a compact mapping, which completes the proof.

2. Spaces with a weak G_δ-diagonal.

A space X is said to have a *weak G_δ-diagonal* if there exists a sequence $\{\mathcal{G}_n\}$ of collections of open subsets of X such that for any pair of distinct points x, y of X there is an n satisfying $\mathrm{St}(x, \mathcal{G}_n) \neq \phi$ and $y \notin \mathrm{St}(x, \mathcal{G}_n)$. A space X is *quasi-developable* if there exists a sequence $\{\mathcal{G}_n\}$ of collections of open subsets of X such that for any $x \in X$ and any open set U containing x there is an n satisfying $\phi \neq \mathrm{St}(x, \mathcal{G}_n) \subset U$ (cf. [4]). A space X has a *θ-T_1-cover* if there exists a sequence $\{\mathcal{V}_n\}$ of collections of open subsets of X such that for any pair of distinct points x, y of X, there is an n such that a) x is in at most finite elements of $\mathcal{V}_n$ and b) $x \in V$ and $y \notin V$ for some $V \in \mathcal{V}_n$ (cf. [11]).

Proposition 1. *If a T_1-space X satisfies one of the following conditions, then X has a weak G_δ-diagonal.*

i) *X has a G_δ-diagonal,*

ii) *X is quasi-developable,*

iii) *X has a θ-T_1-cover (especially, a θ-base).*

Proof. The proof for the case i) is evident by a theorem due to Ceder [7]. In case ii), since X is a quasi-developable T_1-space, X has a weak G_δ-diagonal. In case iii), X has a θ-T_1-cover $\{\mathcal{V}_i\}$. For each n and k, we set $\mathcal{U}_{nk}=\left\{U : U=\bigcap_{i=1}^{k} V_{\alpha_i}\neq\phi,\ V_{\alpha_i}\text{'s are distinct elements of } \mathcal{V}_n\right\}$. If we order $\{\mathcal{U}_{nk} : n, k=1, 2, \cdots\}$ into a sequence, it can easily be seen that X has a weak G_δ-diagonal, which completes the proof.

Proposition 2. *If a space X has a weak G_δ-diagonal and each closed set of X is a G_δ, then X has a G_δ-diagonal.*

Proof. Since X has a weak G_δ-diagonal, there is a sequence $\{\mathcal{G}_n\}$ by the definition. Let $G_n=\cup\{G : G \in \mathcal{G}_n\}$ and $G_n=\bigcup_{i=1}^{\infty} F_{ni}$, where F_{ni} is a closed set of X for each i. If we set $\mathcal{U}_{ni}=\mathcal{G}_n \cup \{X-F_{ni}\}$ and order $\{\mathcal{U}_{ni} : n, i=1, 2, \cdots\}$ into a sequence, then it is shown that X has a G_δ-diagonal by this sequence, which completes the proof.

The following Theorem 2 is a generalization of [6, Proposition 2.9] and [4, Theorem 1].

Theorem 2. *A regular space X has a uniform base if and only if X is a metacompact $w\Delta$-space with a weak G_δ-diagonal.*

Proof. Since the proof of 'only if' part is evident, we prove 'if' part. If each closed set of X is shown to be a G_δ, then X has a G_δ-diagonal, by Proposition 2. Then X has a uniform base by [16, Corollary 5.5]. Therefore we prove that each closed set F of X is a G_δ. We can assume that F has no isolated point. There is a sequence $\{\mathcal{G}_n\}$ of open collections of X by the definition of a weak G_δ-diagonal. Since for each $x \in F$, $\{n : \mathrm{St}(x, \mathcal{G}_n)\neq\phi\}$ is infinite, it is denoted by $\{x(i) : i=1, 2, \cdots\}$, where $x(i)<x(i+1)$ for each i. We take a set $G(x, x(i)) \in \mathcal{G}_{x(i)}$ for each i such that $x \in G(x, x(i))$. Since X is a $w\Delta$-space, there is a sequence $\{\mathcal{B}_n\}$ of open coverings of X satisfying the (M)-condition of K. Morita [14]. We take a $B_i(x) \in \mathcal{B}_i$ such that $x \in B_i(x)$ for each i. Let $U_1(x)=G(x, x(1)) \cap B_1(x)$ and let $\mathcal{U}_1=\{U_1(x) : x \in F\}$. Since X is metacompact, we have a point-finite open collection $\mathcal{V}_1$ of X which refines $\mathcal{U}_1$ and covers F. Since X is regular, there is an open neighborhood $U_2(x)$ of x such that $\overline{U_2(x)}\subset\bigcap_{j=1}^{2} G(x, x(j)) \cap B_2(x) \cap C(x, \mathcal{V}_1)$, where $C(x,$

$\mathcal{V}_1)=\cap\{V: x\in V\in\mathcal{V}_1\}$. Let $\mathcal{U}_2=\{U_2(x): x\in F\}$. Then we have an open, point-finite collection $\mathcal{V}_2$ of X which refines $\mathcal{U}_2$ and covers F. Then there is an open neighborhood $U_3(x)$ of x such that $\overline{U_3(x)}\subset\bigcap_{j=1}^{3} G(x, x(j))\cap B_3(x)\cap C(x,\mathcal{V}_2)$. Let $\mathcal{U}_3=\{U_3(x): x\in F\}$. We repeat this procedure and obtain open, point-finite collections $\{\mathcal{V}_i\}$ of X such that $\mathcal{V}_i$ refines $\mathcal{U}_i$ and covers F. If we set $V_i=\bigcup_{V\in\mathcal{V}_i} V$, then it will be shown that $F=\bigcap_{i=1}^{\infty}V_i$ by an analogous method to the proof of [4, Theorem 1]. Suppose on the contrary. We have a point $y\in\bigcap_i V_i-F$. For each $V\in\mathcal{V}_i$ such that $y\in V$, we have that $V\subset U_i(x_V)\in\mathcal{U}_i$ for some $x_V\in F$. Hence $\mathcal{U}_i(y)=\{U_i(x_V): y\in V\in\mathcal{V}_i\}$ $(i=1,2,\cdots)$ is a finite subcollection of $\mathcal{U}_i$ such that the closure of each element of $\mathcal{U}_{i+1}(y)$ is a subset of some element of $\mathcal{U}_i(y)$ for each i. Then there is $\{U_i(x_i): i=1,2,\cdots\}$ such that $U_i(x_i)\in\mathcal{U}_i(y)$ and $x_i\in F$ and $\overline{U_{i+1}(x_{i+1})}\subset U_i(x_i)$, by [13, Theorem 114]. Since $y\in U_i(x_i)\subset B_i(x_i)$ for each i, the sequence $\{x_i\}$ has a cluster point $x\in F$. Hence $x\neq y$. Then there is an n_1 such that $y\notin \mathrm{St}\,(x,\mathcal{G}_{n_1})\neq\phi$. Since x is a cluster point of $\{x_i\}$, there is a $k>n_1$ such that $x_k\in\mathrm{St}\,(x,\mathcal{G}_{n_1})$. Since $x\in\bigcap_i\overline{U_i(x_i)}=\bigcap_i U_i(x_i)$, we have $x\in U_k(x_k)\subset\bigcap_{j=1}^{k} G(x_k, x_k(j))$, where $n_1<k\leq x_k(k)$. Then we have that $y\in U_k(x_k)\subset G(x_k, n_1)\subset\mathrm{St}\,(x,\mathcal{G}_{n_1})$, which is a contradiction. This implies that each closed set of X is a G_δ, and hence we complete the proof.

The following theorem is a generalization of the well-known metrization theorem due to Okuyama-Borges.

Theorem 3. *A Hausdorff space X is metrizable if and only if X is a paracompact M-space with a weak G_δ-diagonal.*

Proof. Necessity is obvious. Sufficiency. By Theorem 2, X is a paracompact developable space. Hence X is metrizable.

Corollary 4. *A compact Hausdorff space with a weak G_δ-diagonal is metrizable.*

3. Spaces with a σ-locally countable base. A space X is *weakly θ-refinable* if each open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}=\bigcup_i\mathcal{V}_i$ such that if $x\in X$, there is an n such that $\{V\in\mathcal{V}_n: x\in V\}$ is nonempty and finite (cf. [5]). A base $\mathcal{B}$ of a space X is a *θ-base* if $\mathcal{B}=\bigcup_i\mathcal{B}_n$, where each $\mathcal{B}_n$ is an open collection of X, and for each $x\in X$ and each open neighborhood U of x, there is an n such that x is in at most finite members of $\mathcal{B}_n$ and $x\in B\subset U$ for some $B\in\mathcal{B}_n$ (cf. [17]).

Fedorčuk [10] proved that a Hausdorff space X is metrizable if and only if X is paracompact and has a σ-locally countable base.

Theorem 5. *If a space X has a σ-locally countable base, then the following are true:*

i) *if X is weakly θ-refinable, then X has a θ-base,*

ii) *if X is metacompact, then X has a σ-point-finite base.*

Proof. ii) is proved by an analogous method to the proof of [10, Theorem 3]. Let us prove i). Let $\bigcup_n \mathcal{B}_n$ be a σ-locally countable base, and for each n and $x \in X$, let $U_n(x)$ be an open neighborhood of x such that $U_n(x)$ meets at most countable elemets of $\mathcal{B}_n$. Then $\{U_n(x) : x \in X\}$ has an open refinement $\bigcup_k \mathcal{V}_{nk}$ by the definition of the weak θ-refinability. Let $\mathcal{V}_{nk} = \{V_{nk\alpha} : \alpha \in \Omega_{nk}\}$ and for each r of natural numbers let $\mathcal{G}_{nkr} = \left\{G : G = \bigcap_{i=1}^{r} V_{nk\alpha_i} \neq \phi,\ V_{nk\alpha_i}\text{'s are distinct } r \text{ elements of } \mathcal{V}_{nk}\right\}$. For each $G \in \mathcal{G}_{nkr}$, we can write $B_j(G)$, $j=1,2,\cdots$, all the elements of $\mathcal{B}_n$ which meet G, and we set $\mathcal{G}_{nkrj} = \{G \cap B_j(G) : G \in \mathcal{G}_{nkr}\}$. If we order $\{\mathcal{G}_{nkrj} : n,k,r,j=1,2,\cdots\}$ into a sequence, then it is seen to be a θ-base of X as follows: If for any $x \in X$ and any open neighborhood U of x, there is an n such that $x \in B \subset U$ for some $B \in \mathcal{B}_n$. Then there is a k such that x is in at most finite members, for instance $V_{nk\alpha_i}$ $(i=1,2,\cdots,r)$, of $\mathcal{V}_{nk}$. Therefore we have $x \in \bigcap_{i=1}^{r} V_{nk\alpha_i} = G \in \mathcal{G}_{nkr}$. Since $B = B_j(G)$ for some j, we have that $x \in G \cap B_j(G) \subset U$, where $G \cap B_j(G) \in \mathcal{G}_{nkrj}$. Since the element of $\mathcal{G}_{nkrj}$ which contains x is one and only one, it has been proved that $\{\mathcal{G}_{nkrj}\}$ is a θ-base of X. This completes the proof.

Corollary 6. *Let X be a metacompact space with a σ-locaally countable base. If X is a $w\Delta$-space or a $\Sigma^{\#}$-space, then X has a uniform base.*

Proof. By Theorem 5 ii), X has a σ-point-finite base. Hence X has a uniform base by Theorem 1.

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