228. Markov Semigroups with Simplest Interaction. II

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We have defined the semigroup with simplest interaction in Part I. In this Part II, we give the definition of the Markov processes with simplest interaction, their decompositions and constructions, and finally our main result, the relation to the branching Markov processes.

Q is always assumed to be a compact Hausdorff space with a countable basis. We employ the notation of Part I.

§1. Definition.

1.0. Let $\bar{X}=(\bar{P}_x, \bar{X}_t)$ be a Markov process with state space $Q_* \cup \{\Delta\}$ whose transition semigroup $(\bar{T}_t)_{t\geqq 0}$ maps $C_0(Q_*)$ into itself and is strongly continuous. Here $Q_* \cup \{\Delta\}$ is the one point compactification of the locally compact space Q_* and, for any element ϕ in $C_0(Q_*)$, we set $\phi(\Delta)=0$.

1.1. Definition. The process $\bar{X}$ is called a *Markov process with simplest interaction* (or briefly, *process with interaction*) if its transition semigroup $(\bar{T}_t)_{t\geqq 0}$ is a semigroup with simplest interaction on $C_0(Q_*)$.

The process $\bar{X}$ is assumed in this paper to be a Hunt process. Since the non-interaction part $(\bar{T}_t^0)_{t\geqq 0}$ of $(\bar{T}_t)_{t\geqq 0}$ constructed in §3 of Part I is dominated by the latter, it is a transition semigroup of some subprocess $(\bar{P}_x, \bar{X}_t^0)$ of $(\bar{P}_x, \bar{X}_t)$; in fact, setting

$$R(\omega)=\inf\{t : \bar{X}_t(\omega) \notin Q_n\} \qquad \text{if } \bar{X}_0(\omega) \in Q_n,\ n\geqq 1, \tag{1}$$

it follows from Theorem 3.1. of Part I that

$$\bar{T}_t^0\phi(\bar{x})=\bar{E}_x[\phi(\bar{X}_t)1_{\{t<R\}}] \qquad (t\geqq 0, \bar{x} \in Q_*) \tag{2}$$

for any $\phi \in C_0(Q_*)$ where 1_A is the indicator function of the set A.

1.2. Definition. The Markov time R is called *first interacting time*. The Markov process $(\bar{P}_X, \bar{X}_t^0)$ is called *non-interacting part*.

Let $X=(P_x, X_t)$ be the Markov process obtained by piecing together the process $\bar{X}^0$. We suppose that X is conservative, which is possible if we assume that the constant functions belong to the domain $\mathcal{D}(\bar{A}^0)$ of the infinitesimal generator $\bar{A}^0$ of $(\bar{T}_t^0)_{t\geqq 0}$ in the Hille-Yosida sense. Since $(\bar{T}_t^0)$ is degenerated, so is the transition semigroup $(T_t)_{t\geqq 0}$ of the process X. It is easy to verify that X is equivalent to an n independent copies of some Hunt process $x=(P_x, x_t)$ with state space Q if $X_0 \in Q_n$ and $n\geqq 1$.

1.3. Definition. The process x is called *base process* of the process with interaction X. The space Q is referred to the *base space*.

We note that x is nothing but the process obtained by restricting the state space of X to Q and that the process X itself is also referred to the base process.

1.4. We introduce several assumptions:

(0) $P_{\bar{x}}(R<+\infty)=1$ *for all* $\bar{x}\in Q_*$.

(I) *The constant functions belong to* $\mathcal{D}(\bar{A}^0)$.

(II) *The limit*

$$\Pi 1_{Q_1}(\bar{x})=\lim_{t\to 0}\frac{1}{t}\bar{T}_t 1_{Q_1}(\bar{x})$$

exists and is finite at each point $\bar{x}\in Q_*\setminus Q_1$.

(III) *There exists a kernel* π *from* Q_* *into itself such that*

$$P_{\bar{x}}(X_R\in\bar{E}\,|\,X_{R-}=\bar{y})=\pi(\bar{y},\bar{E})$$

for any $\bar{x}$ *and* $\bar{y}$ *in* Q_* *and Borel subset* $\bar{E}$ *of* Q_*.

1.5. Remark. a) The kernel π is called the *interaction law* if it exists.

b) It is easy to see that (I) implies (0) and that (II) implies (III) under the assumption (I).

§2. Decomposition.

2.1. Theorem. *Under the assumptions* (I) *and* (III), *the process with interaction* $\bar{X}$ *is equivalent to the process obtained by piecing together the* $\exp\left(-\int_0^t \bar{q}(X_s)ds\right)$*-subprocess* X^0 *of the base process* X *by the kernel* π.

Moreover if we set $\Pi=q\pi$, *then it is a derivation*:

(3) $$\Pi(\phi*\psi)=(\Pi\phi)*\psi+\phi*(\Pi\phi)\qquad(\phi,\psi\in C(Q_*))$$

2.2. The above theorem is the immediate consequence of the following lemmas.

Lemma. *Let* $\bar{q}$ *be a continuous function on* Q_*. *The multiplication operator* $\bar{q}\cdot$ *is a derivation, i.e.,*

(4) $$\bar{q}\cdot(\phi*\psi)=(\bar{q}\cdot\phi)*\psi+\phi*(\bar{q}\cdot\psi)$$

holds for any ϕ *and* ψ *in* $C(Q_*)$, *if and only if there exists a continuous function* q *on* Q *such that*

(5) $$\bar{q}(x_1,\cdots,x_n)=q(x_1)+\cdots+q(x_n)$$

for any $n\geqq 1$ *and* $x_1,\cdots,x_n\in Q$.

2.3. Corollary. *Under the assumption* (I), $\bar{X}^0$ *is the* $\exp\left(-\int_0^t\bar{q}(X_s)ds\right)$*-subprocess of* X *with* $\bar{q}=-\bar{A}^0 1\in C_0(Q_*)$ *and* (5) *holds for some* q.

2.4. Remark. In general, $R=R(\omega)$ is of the form $\min_{1\leqq j\leqq n} R_0(\omega_j)$ for some Markov time R_0 of x if we set $X_t(\omega)=(x_t(\omega_1),\cdots,x_t(\omega_n))$ where $\{x_t(\omega_i): 1\leqq i\leqq n\}$ denote n independent copies of x.

Furthermore, we can prove that the processes with interaction are invariant under the transformation by the multiplicative functional M_t which is given by

$$M_t(\omega)=M_t(\omega^1)\cdots M_t(\omega^n)$$

2.5. Lemma. *Let $\phi\in C(Q_*)$. Under the assumption* (0), *the limit*

$$\lim_{t\to 0}\frac{1}{t}(\bar{T}_t-\bar{T}_t^0)\phi \tag{6}$$

exists in the topology of uniform or simple convergence on $C_0(Q_*)$ if and only if

$$\lim_{t\to 0}\frac{1}{t}(\bar{T}_t^0-I)K\phi \tag{7}$$

exists in that topology, and the two limits coincide, where

$$K\phi(\bar{x})=E_{\bar{x}}[\phi(X_R)]. \tag{8}$$

We denote the common limit by $\Pi\phi$ if it exists, and, by $\mathfrak{D}(\Pi)$ and $\overline{\mathfrak{D}}(\Pi)$, the collection of ϕ's for which the limit exists in the sense of uniform and simple convergence, respectively.

2.6. Corollary. *Under the assumption* (II), *the operator Π is defined by a nonnegative proper kernel from Q_* into Q_* such that, for each $n\geqq 1$,*

$$\Pi\Big(\cdot, \bigcup_{m\geqq n} Q_m\Big)=0 \qquad on\ Q_n.$$

2.7. Proposition. *If ϕ and ψ are in $\overline{\mathfrak{D}}(\Pi)$ and if $\phi*\psi$ belongs to $C_0(Q_*)$, then $\phi*\psi$ is also in $\overline{\mathfrak{D}}(\Pi)$ and satisfies the relation* (3).

2.8. Remark. From this proposition it follows that Π is completely determined if the information of $\Pi(\cdot, E)$ for $E\subset Q$ is given.

§ 3. Construction.

3.0. Let $\boldsymbol{x}=(P_x, x_t)$ be a given conservative Feller process with state space Q, q a continuous function on Q, and π a substochastic kernel from Q_* into Q. We define the function $\bar{q}$ on Q_* by the relation (5), the kernel $\bar{\pi}$ by (3) with $\Pi=\bar{q}\bar{\pi}$, and $\boldsymbol{X}=(P_x, X_t)$ to be the process with state space Q_* which is equivalent to n independent copies of $\boldsymbol{x}$ on each Q_n.

3.1. Theorem. *For a given system $(\boldsymbol{x}, q, \pi)$, there exists a unique Feller process with interaction $\boldsymbol{X}=(\bar{P}_x, \bar{X}_t)$ satisfying the following three properties:*

1) *its base process is $\boldsymbol{x}$*

2) *its first interaction time is distributed according to the law $\exp\left(-\int_0^t \bar{q}(X_s)ds\right)$ if it is conditioned with the non-interacting part.*

3) *its interaction law is $\bar{\pi}$.*

3.2. Remark. The transition semigroup $(\bar{T}_t)_{t\geqq 0}$ of $\overline{X}$ is the unique solution of the following integral equation:

$$\bar{T}_t = \bar{T}_t^0 + \int_0^t ds \bar{T}_s^0 \Pi \bar{T}_{t-s} \tag{9}$$

where $\Pi = \bar{q}\bar{\pi}$ and $(\bar{T}_t^0)_{t \geqq 0}$ is the transition semigroup of the $\exp\left(-\int_0^t \bar{q}(X_s)ds\right)$-subprocess $\bar{X}^0$ of X, or equivalently to say, the semigroup whose infinitesimal generator $\bar{A}^0$ is given by

$$\bar{A}^0 = A - \bar{q}\cdot$$

if A denotes the infinitesimal generator of X.

3.3. Sketch of the proof. It suffices to solve the equation (9) and to show that the unique solution $(\bar{T}_t)_{t \geqq 0}$ satisfies the interaction property. This can be done by a usual successive approximation. Put, inductively in $n \geqq 0$,

$$\bar{T}_t^{n+1}\phi(\bar{x}) = \bar{E}_{\bar{x}}^0 \left[S \leqq t, \int_{Q_*} \pi(\bar{X}(S-), d\bar{y}) \bar{T}_t^n \phi(\bar{y}) \right] + \bar{T}_t^0 \phi(\bar{x}) \tag{10}$$

for $\phi \geqq 0$ in $C_0(Q_*)$ where S is the life time of $\bar{X}^0$. Then $\bar{T}_t^n$ is nondecreasing and converges to some $\bar{T}_t$. The uniqueness is immediate from the relation (3). The interaction property is a consequence of the property:

$$\bar{T}_t^n(\phi * \psi) = \sum_{i+j=n} (\bar{T}_t^i \phi) * (\bar{T}_t^j \psi) \tag{11}$$

for any ϕ and ψ.

3.4. Example. Let π_n be a substochastic kernel from Q^n into Q and q_n a continuous function on Q for each $n \geqq 2$. Suppose that $q = \sum_{n \geqq 2} q_n$ is bounded. The nonlinear equation of the following type for substochastic measures has been studied by [2] [3] [7] etc.

$$\begin{aligned} \frac{du}{dt}(t, E) &= Bu(t, E) \\ &\quad + \sum_{n \geqq 2} \int_{Q^n} u(t, dx_1) \cdots u(t, dx_n) q_u(x_1)(\pi_1(x_1, \cdots, x_n; E) - \delta_{x_1}(E)) \end{aligned} \tag{12}$$

where B is a (dual) generator of some Markov process X with state space Q.

This class of nonlinear equation can be linearized if we construct the Markov process with state space Q_*. In fact, noting that π_n may be assumed to be symmetric in $x_1, \cdots, x_n$, if we put

$$\pi(\bar{x}, E) = \frac{1}{n} \frac{q_n(\bar{x})}{q(\bar{x})} \pi_n(\bar{x}, E) \tag{13}$$

for $\bar{x} \in Q_*$ and $E \subset Q$ and then extend it by (3) with $\Pi = q\pi$, then the system (x, q, π) satisfies the conditions of the theorem.

This idea of linearization can be applied to a class of nonlinear evolution equations, for example, the Burgers' equation for which the "interaction law" is a differential operator. The details will be published elsewhere.

§ 4. Duality with branching processes.

4.0. It is intuitively feasible that the reversed process of a process with interaction might be a branching process. Indeed, it is true for the case where Q is a one-point space (See 4.7). In general we have the following theorem.

4.1. Theorem. *Let $(\bar{T}_t)_{t\geqq 0}$ be the transition semigroup of a Hunt process $\bar{X}=(P_{\bar{x}}, X_t)$ with state space Q_* whose transition kernel $P(t, \bar{x}, \bar{E})$ exists. Suppose that there is a nonnegative Borel measure μ_0 on Q_* of the form M^*m_0 for some measure m on Q such that*

1) $T_t^*\mu_0 \leqq \mu_0$ *for any* $t\geqq 0$,

2) *the measure $P(t, \bar{x}, \cdot)$ is absolutely continuous with respect to μ_0 for each $t>0$ and $\bar{x}\in Q_*$,*

3) *the function*

$$\bar{x}\to P(t, \bar{x}, \bar{y})\equiv P(t, \bar{x}, d\bar{y})/\mu_0(d\bar{y})$$

is continuous on Q_* for each $t>0$ and $\bar{x}\in Q_*$.

Let $(\hat{T}_t)_{t\geqq 0}$ be the dual semigroup of $(T_t)_{t\geqq 0}$ with respect to μ_0;

$$\int(\hat{T}_t\phi)(\bar{x})\psi(\bar{x})\mu_0(d\bar{x})=\int\phi(\bar{x})(T_t\psi)(\bar{x})\mu_0(d\bar{x}). \tag{14}$$

Then, (i) *$(\hat{T}_t)_{t\geqq 0}$ is a branching Markov semigroup if X is a process with interaction.*

(ii) *Conversely, if X is a branching process, then $(\hat{T}_t)_{t\geqq 0}$ is a interaction Markov semigroup.*

4.2. Remark. a) The condition 3) is merely a regularity condition imposed by our formulation in $C_0(Q_*)$.

b) The assumption 1) holds if μ_0 is (T_t^*)-invariant, or equivalently, if μ_0 satisfies the condition $\langle\mu_0, A\phi\rangle=0$ for any ϕ in $\mathcal{D}(A)$. This is the case for the homogeneous Boltzmann equations with m_0 Gaussian distributions. In particular, if the base process is trivial, i.e., if $x_t\equiv x_0$, the condition is equivalent to the following:[1]

$$\sum_{n\geqq 2}\int_{Q^n} m_0(dx_1)\cdots m_0(dx_n)q(x_1)\pi(x_1, \cdots, x_n, E)=\int_E m_0(dx)q(x) \tag{15}$$

provided that there exist the interaction law π and the interacting rate q.

c) This theorem is already used implicitly in the work of S. Tanaka [6].

4.3. Proof. Define a map J from $C_0(Q_*)$ into $\mathcal{M}(Q_*)$ by

$$J\phi(d\bar{x})=\phi(\bar{x})\mu_0(d\bar{x}) \qquad (\phi\in C_0(Q_*))$$

and a map J_1 from $C(Q)$ into $\mathcal{M}(Q)$ by

$$J_1 f(dx)=f(x)m_0(dx) \qquad (f\in C(Q)).$$

The inverse maps J^{-1} and J_1^{-1} of J and J_1 are defined respectively on the range $\mathcal{D}$ and $\mathcal{D}_1$ of the latters. From the assumptions, it follows that $T_t^*(\mathcal{M}(Q_*))\subset\mathcal{D}$ and the following properties:

1) This case has been studied in detail by T. Ueno (unpublished).

(a) $JM = M^*J_1 = MR_1JM$

(b) $J^{-1}M^* = M^*J_1^{-1} = M^*R_1^*J^{-1}M^*$

(c) J and J^{-1} are algebra homomorphisms (from $\mathcal{C}(Q_*)$ into $\mathcal{M}(Q_*)$ and from $\mathcal{D}$ into $\mathcal{M}(Q_*)$ respectively).

Noting that $\hat{T}_t = J^{-1}T_t^*J$, the theorem follows from an operational calculus.

4.4. On the reversed process we have the following:

Proposition. *Let $(P_\mu, \bar{X}_t)$ be a Feller process with interaction such that $\sup_{x \in Q} E_x[R] < +\infty$. Then the reversed process at any L-time*[2] *is a Markov process and its transition semigroup is in duality with that of (P_μ, X_t) relative to the measure μV where V is the potential operator for (P_μ, X_t).*

The existence of the operator V is justified by the next lemma.

4.5. Lemma. *Let $(T_t)_{t \geq 0}$ be the transition semigroup of a Feller process with interaction. Then the following integral exists and is finite for each $\phi \in \mathcal{C}_0(Q_*)$ and $\bar{x} \in Q_*$.*

(16) $$V\phi(\bar{x}) \equiv \int_0^\infty T_t\phi(\bar{x})dt$$

and satisfies the relation

(17) $$\sup_{x \in Q_p} |V\phi(\bar{x})| \leq \|\phi\| \left(\sum_{k=1}^{p} \frac{1}{k}\right) \sup_{x \in Q} E_x[R].$$

4.6. Remark. a) The potential operators V of interaction semigroups are characterized in $\mathcal{C}(Q_*)$ by the relation

(18) $$(V\phi) * (V\psi) = V[(V\phi) * \psi + \phi * (V\psi)].$$

b) Under the assumption of 4.4, $T_t\phi(\bar{x})$ tends to zero as $t \to \infty$ at each point $\bar{x} \in Q_*$ and $\phi \in \mathcal{C}(Q_*)$. If m is a substochastic measure on Q with $m(Q) < 1$, then the total mass of the measure $R^*T_t^*M^*m$ tends to zero as $t \to \infty$.

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References

[1]–[4] Cf. References at the end of Part I (p. 978).

[5] Nagasawa, M.: Time reversion of Markov processes. Nagoya Math. J., **24** (1964).

[6] Tanaka, S.: An extension of Wild's sum for solving certain non-linear equation of measures. Proc. Japan Acad., **44** (1969).

[7] Ueno, T.: A class of Markov processes with interaction. I, II. Proc. Japan Acad., **45** (1969).

2) See [5].