211. *A Perturbation Theorem for Linear Contraction Semigroups on Reflexive Banach Spaces*

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1. Introduction and statement of the result. This paper is concerned with the perturbation of linear contraction semigroups on Banach spaces. Our result gives a partial extension of a perturbation theorem for such semigroups obtained by Nelson [5] and Gustafson [1].

A linear operator A (with domain $D(A)$ and range $R(A)$) in an arbitrary Banach space X is said to be *accretive* if

(A) $\quad \|(A+\xi)u\| \geqq \xi\,\|u\| \quad$ for every $u \in D(A)$ and $\xi > 0$.

This implies that $(A+\xi)^{-1}$ exists and $\|(A+\xi)^{-1}v\| \leqq \xi^{-1}\|v\|$, $v \in R(A+\xi)$, for $\xi>0$. It can be shown that $(A+\xi)^{-1}$ has domain X either for every $\xi>0$ or for no $\xi>0$; in the former case we say that A is *m-accretive*. Also, (A) is equivalent to the following condition:

(A′) For every $u \in D(A)$ there is $f \in F(u)$ such that

$$\mathrm{Re}\,(Au, f) \geqq 0,$$

where F denotes the duality mapping: $F(u)=\{f \in X^*;\,(u,f)=\|u\|^2=\|f\|^2\}$ (cf. Kato [3] in which the term "monotonic" was used instead of "accretive"). Note that the inequality is not required to hold for every $f \in F(u)$. But if X is *reflexive* and A is m-accretive, then the inequality holds for every $f \in F(u)$. This is a consequence of the following facts (cf. Lumer-Phillips [4], Remark 1 to Theorem 3.1):

1) $-A$ is the (infinitesimal) generator of a linear contraction semigroup on an arbitrary Banach space if and only if A is m-accretive and densely defined;

2) If X is reflexive, then m-accretive operators in X are necessarily densely defined (cf. Kato [2], or Yosida [9], p. 218).

In fact, being the generator of a linear contraction semigroup is independent of the multiplicity of duality mapping.

On the (relatively bounded) perturbation of linear contraction semigroups, we know the following result due to Nelson and Gustafson (cf. [1]):

Theorem 1. *Let A and B be linear operators in an arbitrary Banach space X such that*

$$\|Bu\| \leqq a\,\|u\| + b\,\|Au\|, \quad a \geqq 0, \quad 0 < b < 1, \quad u \in D(A) \subset D(B).$$

If $-A$ is the generator of a linear contraction semigroup and if B is accretive, then $-(A+B)$ defined on $D(A)$ is also the generator of such a semigroup.

The purpose of the present paper is to prove the following

Theorem 2. *Let A and B be linear operators in a reflexive Banach space X such that*

(1) $$\|Bu\| \leqq a\|u\| + \|Au\|, \quad a \geqq 0, \quad u \in D(A) \subset D(B).$$

If A is m-accretive and B is accretive, then the closure of $A+B$ defined on $D(A)$ is also m-accretive.

Theorem 2 is a generalization of a perturbation theorem proved recently by Wüst [8]. The proof of Theorem 2 is based on Theorem 1.

2. Proof of Theorem 2. Since X is reflexive and A is m-accretive, we see that $A+B$ is accretive and densely defined. Hence, $A+B$ is closable (cf. [4], Lemma 3.3). Since the closure C of $A+B$ is also accretive, it suffices to show that the range $R(C+1)$ of $C+1$ is equal to the whole space X.

Since $\|tBu\| \leqq a\|u\| + t\|Au\|$ for $0<t<1$ and $u \in D(A) \subset D(B)$, it follows from Theorem 1 that $A+tB$ is m-accretive for $t \in [0,1)$. This means that $(A+tB+1)D(A)=X$ for $t \in [0,1)$, that is, for any $w \in X$ there exists a family $\{u(t) \in D(A)\,;\, 0 \leqq t < 1\}$ such that

(2) $$w = (A+tB+1)u(t), \qquad t \in [0,1).$$

To obtain $R(C+1)=X$, we shall show that w belongs to $R(C+1)$. Setting $v(t)=(A+B+1)u(t)$, we have

(3) $$v(t)-w=(1-t)Bu(t), \qquad t \in [0,1).$$

Also, we see from (2) that

(4) $$\|u(t)\| \leqq \|(A+tB+1)^{-1}\|\,\|w\| \leqq \|w\|, \qquad t \in [0,1).$$

It then follows from (3), (1) and (4) that

$$\begin{aligned}
\|v(t)\| &\leqq \|v(t)-w\| + \|w\| = (1-t)\|Bu(t)\| + \|w\| \\
&\leqq a\|u(t)\| + \|Au(t)\| - t\|Bu(t)\| + \|w\| \\
&\leqq (a+1)\|w\| + \|(A+tB)u(t)\| \\
&\leqq (a+1)\|w\| + \|(A+tB+1)u(t)\| + \|u(t)\| \\
&\leqq (a+3)\|w\|.
\end{aligned}$$

Thus, the family $\{v(t) \in X\,;\, 0 \leqq t < 1\}$ is bounded. Since X is locally sequentially weakly compact (the Eberlein-Shmulyan theorem), we can find a sequence $\{t_n\} \subset [0,1)$ and an element $v_1 \in X$ such that $t_n \to 1$ and $\{v(t_n)\} \subset R(C+1)$ converges weakly to v_1 as $n \to \infty$. But since $\|(C+1)^{-1}v\| \leqq \|v\|$, $v \in R(C+1)$, by condition (A), it follows from the closedness of C that $R(C+1)$ is a closed linear subspace of X. Thus, $R(C+1)$ is weakly closed and we have $v_1 \in R(C+1)$. Consequently, to see that $w \in R(C+1)$, it suffices to show that $v_1 = w$.

Now let B^* be the adjoint operator of B. Then we have by (3) that for $g \in D(B^*)$,

$$|(v_1-w, g)|=\lim_{n\to\infty}|(v(t_n)-w, g)|=\lim_{n\to\infty}|((1-t_n)Bu(t_n), g)|$$
$$=\lim_{n\to\infty}(1-t_n)|(u(t_n), B^*g)|$$
$$\leqq\|w\|\,\|B^*g\|\lim_{n\to\infty}(1-t_n)=0.$$

Since X is reflexive and B is closable, B^* is densely defined. Thus we obtain $(v_1-w, g)=0$ for any $g\in X^*$ and hence $w=v_1\in R(C+1)$. This completes the proof of Theorem 2.

3. Remarks.

Remark 1. Let A and B be linear operators in a reflexive Banach space X such that $\|Bu\|^2\leqq a'^2\|u\|^2+\|Au\|^2$, $a'>0$, $u\in D(A)\subset D(B)$. If A is m-accretive and B is accretive, then the closure of $A+B$ defined on $D(A)$ is also m-accretive. In fact, since $\|Bu\|\leqq(a'^2\|u\|^2+\|Au\|^2)^{1/2}\leqq a'\|u\|+\|Au\|$, we can apply the result obtained above. When X is a Hilbert space, this fact is noted in [6].

Remark 2. Theorem 2 may be false if X is not reflexive. In fact, Trotter [7] has given an example of two densely defined m-accretive operators A, B in $X=C[-\infty, \infty]$ such that $\|Bu\|\leqq\|Au\|$ for $u\in D(A)=D(B)$, but the closure of $A+B$ is not m-accretive.

Remark added in proofs. After this paper was submitted for publication, the writer received a preprint of [10].

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