10. *Probabilities on Inheritance in Consanguineous Families. III*

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(Comm. by T. FURUHATA, M.J.A., Jan. 12, 1954)

III. Simple mother-descendants combinations (Continuation)

3. General mother-descendants combination

The problems in the preceding sections concern a combination consisting of an individual and its two collateral descendants in which a collateral separation takes place at the original generation. We shall now consider a mother-descendants combination in which a collateral separation appears at a certain intermediate generation. In fact, we introduce a probability

$$\pi_{\iota|\mu\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2) \equiv \overline{A}_{\alpha\beta}\kappa_{\iota|\mu\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)$$

which is defined by an equation

$$\kappa_{\iota|\mu\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2) = \sum \kappa_\iota(\alpha\beta;\ ab)\kappa_{\mu\nu}(ab;\ \xi_1\eta_1,\ \xi_2\eta_2).$$

According to three systems for the $\kappa_{\mu\nu}$'s, we distinguish here also *three systems*, i.e. $\mu=\nu=1$, $\mu=1<\nu$ or $\mu>1=\nu$, and $\mu,\nu>1$.

The formula for the lowest system is then expressed in the form

$$\kappa_{\iota|11}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2) = \sigma(\xi_1\eta_1,\ \xi_2\eta_2)+2^{-\iota+1}U(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2),$$

where the quantity U is defined by

$$U(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2) = \sum Q(\alpha\beta;\ ab)\kappa(ab;\ \xi_1\eta_1,\ \xi_2\eta_2).$$

It is symmetric with respect to $\xi_1\eta_1$ and $\xi_2\eta_2$, and its values are listed as follows; cf. a remark stated at the end of I, § 1:

$$U(ii;\ ii,\ ii) = \tfrac{1}{8}i(1-i)(1+i)(1+2i),\quad U(ii;\ ii,\ ig) = \tfrac{1}{4}ig(1-2i^2),$$
$$U(ii;\ ii,\ gg) = \tfrac{1}{8}ig^2(1-2i),\quad U(ii;\ ii,\ fg) = \tfrac{1}{4}ifg(1-2i),$$
$$U(ii;\ ik,\ ik) = \tfrac{1}{8}k(1+k-3i^2+ik-8i^2k),$$
$$U(ii;\ ik,\ kk) = \tfrac{1}{8}k^2(1-3i+k-4ik),$$
$$U(ii;\ ik,\ ig) = \tfrac{1}{8}kg(1+i-8i^2),\quad U(ii;\ ik,\ kg) = \tfrac{1}{8}kg(1-3i+2k-8ik),$$
$$U(ii;\ ik,\ gg) = \tfrac{1}{8}kg^2(1-4i),\quad U(ii;\ ik,\ fg) = \tfrac{1}{4}kfg(1-4i),$$
$$U(ii;\ kk,\ kk) = -\tfrac{1}{8}k^2(1+k)(1+2k),\quad U(ii;\ kk,\ kg) = -\tfrac{1}{8}k^2g(3+4k),$$
$$U(ii;\ kk,\ gg) = -\tfrac{1}{4}k^2g^2,\quad U(ii;\ kk,\ fg) = -\tfrac{1}{2}k^2fg,$$
$$U(ii;\ hk,\ hk) = -\tfrac{1}{8}hk(2+3h+3k+8hk),\quad U(ii;\ hk,\ kg) = -\tfrac{1}{8}hkg(3+8k),$$
$$U(ii;\ hk,\ fg) = -hkfg;$$
$$U(ij;\ ii,\ ii) = \tfrac{1}{16}i(1-2i)(1+i)(1+2i),$$
$$U(ij;\ ii,\ ij) = \tfrac{1}{16}i(i+2j+i^2-3ij-8i^2j),$$
$$U(ij;\ ii,\ jj) = \tfrac{1}{16}ij(i+j-4ij),\quad U(ij;\ ii,\ ig) = \tfrac{1}{16}ig(2-3i-8i^2),$$
$$U(ij;\ ii,\ jg) = \tfrac{1}{16}ig(i+2j-8ij),\quad U(ij;\ ii,\ gg) = \tfrac{1}{16}ig^2(1-4i),$$
$$U(ij;\ ii,\ fg) = \tfrac{1}{8}ifg(1-4i),$$

$U(ij;\, ij,\, ij) = \frac{1}{16}(i+j+i^2+j^2-2ij(i+j+8ij)),$

$U(ij;\, ij,\, ig) = \frac{1}{16}g(i+j+2i^2-2ij-16i^2j),$

$U(ij;\, ij,\, gg) = \frac{1}{16}g^2(i+j-8ij), \quad U(ij;\, ij,\, fg) = \frac{1}{8}fg(i+j-8ij),$

$U(ij;\, ik,\, ik) = \frac{1}{16}k(1-2i+k-6i^2-2ik-16i^2k),$

$U(ij;\, ik,\, jk) = \frac{1}{16}k(i+j-6ij+2ik+2jk-16ijk),$

$U(ij;\, ik,\, kk) = \frac{1}{16}k^2(1-6i+k-8ik), \quad U(ij;\, ik,\, ig) = \frac{1}{16}kg(1-2i-16i^2),$

$U(ij;\, ik,\, jg) = \frac{1}{16}kg(i+j-16ij), \quad U(ij;\, ik,\, kg) = \frac{1}{16}kg(1-6i+2k-16ik),$

$U(ij;\, ik,\, gg) = \frac{1}{16}kg^2(1-8i), \quad U(ij;\, ik,\, fg) = \frac{1}{8}kfg(1-8i).$

It is noted that there hold the identities

$$\sum U(\alpha\beta;\, \xi\eta,\, ab) = \tfrac{1}{2}Q(\alpha\beta;\, \xi\eta), \quad \sum \overline{A}_{ab}U(ab;\, \xi_1\eta_1,\, \xi_2\eta_2) = 0,$$

$$\sum Q(\alpha\beta;\, ab)U(ab;\, \xi_1\eta_1,\, \xi_2\eta_2) = \tfrac{1}{2}U(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2).$$

The formula for the second system is expressed in the form

$$\kappa_{l1\nu}(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2) = \sigma_{1\nu}(\xi_1\eta_1,\, \xi_2\eta_2) + 2^{-l}\,\overline{A}_{\xi_2\eta_2}Q(\alpha\beta;\, \xi_1\eta_1) + 2^{-l-\nu+1}V(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2),$$

where the quantity V is defined by

$$V(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2) = \sum Q(\alpha\beta;\, ab)\,W(ab;\, \xi_1\eta_1,\, \xi_2\eta_2),$$

and its values are listed as follows; cf. again the remark stated at the end of I, § 1:

$V(ii;\, ii,\, ii) = \frac{1}{2}i^2(1-i)(2-i), \quad V(ii;\, ii,\, ig) = \frac{1}{2}ig(1-2i)(2-i),$

$V(ii;\, ii,\, gg) = -\frac{1}{2}ig^2(2-i), \quad V(ii;\, ii,\, fg) = -ifg(2-i),$

$V(ii;\, ik,\, ii) = \frac{1}{2}ik(1-2i+2i^2), \quad V(ii;\, ik,\, ik) = \frac{1}{2}k(i+k-2i^2-4ik+4i^2k),$

$V(ii;\, ik,\, kk) = \frac{1}{2}k^2(1-2i-2k+2ik), \quad V(ii;\, ik,\, ig) = \frac{1}{2}kg(1-2i)^2,$

$V(ii;\, ik,\, kg) = \frac{1}{2}kg(1-2i-4k+4ik), \quad V(ii;\, ik,\, gg) = -kg^2(1-i),$

$V(ii;\, ik,\, fg) = -2kfg(1-i),$

$V(ii;\, kk,\, ii) = \frac{1}{2}ik^2(1+i), \quad V(ii;\, kk,\, ik) = -\frac{1}{2}k^2(2i-k-2ik),$

$V(ii;\, kk,\, kk) = -\frac{1}{2}k^3(2-k), \quad V(ii;\, kk,\, ig) = \frac{1}{2}k^2g(1+2i),$

$V(ii;\, kk,\, kg) = -k^2g(1-k), \quad V(ii;\, kk,\, gg) = \frac{1}{2}k^2g^2,$

$V(ii;\, kk,\, fg) = k^2fg,$

$V(ii;\, hk,\, ii) = ihk(1+i), \quad V(ii;\, hk,\, ik) = -hk(i-k-2ik),$

$V(ii;\, hk,\, kk) = -hk(i-k-2ik), \quad V(ii;\, hk,\, hk) = -hk(h+k-2hk),$

$V(ii;\, hk,\, ig) = hkg(1+2i), \quad V(ii;\, hk,\, kg) = -hkg(1-2k),$

$V(ii;\, hk,\, gg) = hkg^2, \quad V(ii;\, hk,\, fg) = 2hkfg;$

$V(ij;\, ii,\, ii) = \frac{1}{4}i^2(2-i)(1-2i), \quad V(ij;\, ii,\, ij) = \frac{1}{4}i(2j+i^2-7ij+4i^2j),$

$V(ij;\, ii,\, jj) = \frac{1}{4}ij(i-2j+2ij), \quad V(ij;\, ii,\, ig) = \frac{1}{4}ig(2-7i+4i^2),$

$V(ij;\, ii,\, jg) = \frac{1}{4}ig(i-4j+4ij), \quad V(ij;\, ii,\, gg) = -\frac{1}{2}ig^2(1-i),$

$V(ij;\, ii,\, fg) = -ifg(1-i),$

$V(ij;\, ij,\, ii) = \frac{1}{4}i(1-2i)(i+j-2ij), \quad V(ij;\, ij,\, ij) = \frac{1}{4}(i+j-4ij)(i+j-2ij),$

$V(ij;\, ij,\, ig) = \frac{1}{4}g(1-4i)(i+j-2ij), \quad V(ij;\, ij,\, gg) = -\frac{1}{2}g^2(i+j-2ij),$

$V(ij;\, ij,\, fg) = -fg(i+j-2ij),$

$V(ij;\, ik,\, ii) = \frac{1}{4}ik(1-2i)^2, \quad V(ij;\, ik,\, ij) = \frac{1}{4}k(j+2i^2-6ij+8i^2j),$

$V(ij;\, ik,\, jj) = \frac{1}{2}jk(i-j+2ij), \quad V(ij;\, ik,\, ik) = \frac{1}{4}k(i+k-4i^2-6ik+8i^2k),$

$V(ij;\, ik,\, jk) = \frac{1}{4}k(j-4ij+2ik-4jk+8ijk),$

$V(ij;\, ik,\, kk) = \frac{1}{4}k^2(1-4i-2k+4ik),$

$V(ij;\, ik,\, ig) = \frac{1}{4}kg(1-4i)(1-2i),$ $V(ij;\, ik,\, jg) = \frac{1}{2}kg(i-2j+4ij),$

$V(ij;\, ik,\, kg) = \frac{1}{4}kg(1-4i-4k+8ik),$ $V(ij;\, ik,\, gg) = -\frac{1}{2}kg^2(1-2i),$

$V(ij;\, ik,\, fg) = -kfg(1-2i),$

$V(ij;\, kk,\, ii) = \frac{1}{4}ik^2(1+2i),$ $V(ij;\, kk,\, ij) = \frac{1}{4}k^2(i+j+4ij),$

$V(ij;\, kk,\, ik) = -\frac{1}{4}k^2(4i-k-4ik),$ $V(ij;\, kk,\, ig) = \frac{1}{4}k^2g(1+4i),$

$V(ij;\, hk,\, ii) = \frac{1}{2}ihk(1+2i),$ $V(ij;\, hk,\, ij) = \frac{1}{2}hk(i+j+4ij),$

$V(ij;\, hk,\, ik) = -\frac{1}{2}hk(2i-k-4ik),$ $V(ij;\, hk,\, ig) = \frac{1}{2}hkg(1+4i).$

It is noted that there hold the identities

$$\sum V(\alpha\beta;\, \xi\eta,\, ab) = 0, \qquad \sum V(\alpha\beta;\, ab,\, \xi\eta) = 2Q(\alpha\beta;\, \xi\eta),$$

$$\sum \overline{A}_{ab} V(ab;\, \xi_1\eta_1,\, \xi_2\eta_2) = 0,$$

$$\sum Q(\alpha\beta;\, ab) V(ab;\, \xi_1\eta_1,\, \xi_2\eta_2) = \tfrac{1}{2} V(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2).$$

The formula for the last generic system is expressed in the form

$$\kappa_{l\mu\nu}(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2) = \sigma_{\mu\nu}(\xi_1\eta_1,\, \xi_2\eta_2) + 2^{-l+1}\{2^{-\mu}\overline{A}_{\xi_2\eta_2} Q(\alpha\beta;\, \xi_1\eta_1)$$
$$+ 2^{-\nu}\overline{A}_{\xi_1\eta_1} Q(\alpha\beta;\, \xi_2\eta_2)\} + 2^{-l-\lambda+1} S(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2),$$
$$\lambda = \mu+\nu-1,$$

where the quantity S is defined by

$$S(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2) = \sum Q(\alpha\beta;\, ab) T(ab;\, \xi_1\eta_1,\, \xi_2\eta_2),$$

and its values are listed as follows; cf. a routine remark:

$S(ii;\, ii,\, ii) = \frac{1}{2}i^2(1-i)(1-2i),$ $S(ii;\, ii,\, ig) = \frac{1}{2}ig(1-2i)^2,$

$S(ii;\, ii,\, gg) = -\frac{1}{2}ig^2(1-2i),$ $S(ii;\, ii,\, fg) = -ifg(1-2i),$

$S(ii;\, ik,\, ik) = \frac{1}{2}k(k-i^2-5ik+8i^2k),$ $S(ii;\, ik,\, kk) = -\frac{1}{2}k^2(i+k-4ik),$

$S(ii;\, ik,\, ig) = \frac{1}{2}kg(1-5i+8i^2),$ $S(ii;\, ik,\, kg) = -\frac{1}{2}kg(i+2k-8ik),$

$S(ii;\, ik,\, gg) = -\frac{1}{2}kg^2(1-4i),$ $S(ii;\, ik,\, fg) = -kfg(1-4i),$

$S(ii;\, kk,\, kk) = \frac{1}{2}k^3(1-2k),$ $S(ii;\, kk,\, kg) = -\frac{1}{2}k^2g(1-4k),$

$S(ii;\, kk,\, gg) = k^2g^2,$ $S(ii;\, kk,\, fg) = 2k^2fg,$

$S(ii;\, hk,\, hk) = -\frac{1}{2}hk(h+k-8hk),$ $S(ii;\, hk,\, kg) = -\frac{1}{2}hkg(1-8h),$

$S(ii;\, hk,\, fg) = 4hkfg;$

$S(ij;\, ii,\, ii) = \frac{1}{4}i^2(1-2i)^2,$ $S(ij;\, ii,\, ij) = \frac{1}{4}i(j-i^2-5ij+8i^2j),$

$S(ij;\, ii,\, jj) = -\frac{1}{4}ij(i+j-4ij),$ $S(ij;\, ii,\, ig) = \frac{1}{4}ig(1-5i+8i^2),$

$S(ij;\, ii,\, jg) = -\frac{1}{4}ig(i+2j-8ij),$ $S(ij;\, ii,\, gg) = -\frac{1}{4}ig^2(1-4i),$

$S(ij;\, ii,\, fg) = -\frac{1}{2}ifg(1-4i),$

$S(ij;\, ij,\, ij) = \frac{1}{4}(i^2+j^2-6ij(i+j)+16i^2j^2),$

$S(ij;\, ij,\, ig) = \frac{1}{4}g(j-2i^2-6ij+16i^2j),$

$S(ij;\, ij,\, gg) = -\frac{1}{4}g^2(i+j-8ij),$ $S(ij;\, ij,\, fg) = -\frac{1}{2}fg(i+j-8ij),$

$S(ij;\, ik,\, ik) = \frac{1}{4}k(k-2i^2-6ik+16i^2k),$

$S(ij;\, ik,\, jk) = -\frac{1}{2}k(ij+ik+jk-8ijk),$

$S(ij;\, ik,\, kk) = -\frac{1}{4}k^2(2i+k-8ik),$ $S(ij;\, ik,\, ig) = \frac{1}{4}kg(1-6i+16i^2),$

$S(ij;\, ik,\, jg) = -\frac{1}{2}kg(i+j-8ij),$ $S(ij;\, ik,\, kg) = -\frac{1}{2}kg(i+k-8ik),$

$S(ij;\, ik,\, gg) = -\frac{1}{4}kg^2(1-8i),$ $S(ii;\, ik,\, fg) = -\frac{1}{2}kfg(1-8i).$

It is noted that the quantity S satisfies, besides an evident symmetry relation $S(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=S(\alpha\beta;\ \xi_2\eta_2,\ \xi_1\eta_1)$, also the identities

$$\sum S(\alpha\beta;\ \xi\eta,\ ab)=0, \qquad \sum \overline{A}_{ab}S(ab;\ \xi_1\eta_1,\ \xi_2\eta_2)=0,$$
$$\sum Q(\alpha\beta;\ ab)S(ab;\ \xi_1\eta_1,\ \xi_2\eta_2)=\tfrac{1}{2}S(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2).$$

The asymptotic behaviors of $\kappa_{l|\mu\nu}$ as ν (or μ) or l tends to infinity will be obvious. In fact, we get the limit relations

$$\lim_{\nu\to\infty}\kappa_{l|\mu\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\kappa_{l+\mu}(\alpha\beta;\ \xi_1\eta_1)\overline{A}_{\xi_2\eta_2}$$

and

$$\lim_{l\to\infty}\kappa_{l|\mu\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\sigma_{\mu\nu}(\xi_1\eta_1,\ \xi_2\eta_2),$$

which remain valid for any values of l,μ with $l\geqq 1,\ \mu\geqq 1$, and of μ,ν with $\mu\geqq 1,\ \nu\geqq 1$, respectively.