9. *Probabilities on Inheritance in Consanguineous Families. II*

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III. Simple mother-descendants combinations

1. Mother-child-νth descendant combination

We designate, in general, by

$$\pi_{\mu\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2) \equiv \bar{A}_{\alpha\beta}\kappa_{\mu\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)$$

the probability of a combination consisting of an individual $A_{\alpha\beta}$ and its μth and νth collateral descendants $A_{\xi_1\eta_1}$ and $A_{\xi_2\eta_2}$, respectively, originated from the same spouse of $A_{\alpha\beta}$.

Three systems will be distinguished according to $\mu=\nu=1$, $\mu=1<\nu$ or $\mu>1=\nu$, and $\mu,\ \nu>1$. The *lowest system* has already been treated as the probability of mother-children combination[1]

$$\pi(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2) \equiv \bar{A}_{\alpha\beta}\kappa(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2) \qquad (\kappa \equiv \kappa_{11}).$$

In the present section we consider the *second system* while the last system will be postponed into the next section.

Now, based on an evident quasi-symmetry relation

$$\pi_{\mu\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2) = \pi_{\nu\mu}(\alpha\beta;\ \xi_2\eta_2,\ \xi_1\eta_1),$$

it suffices to deal with the former of the second system. The reduced probability $\kappa_{1\nu}$ is then defined by a recurrence equation

$$\kappa_{1\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2) = \sum \kappa(\alpha\beta;\ \xi_1\eta_1,\ ab)\kappa_{\nu-1}(ab;\ \xi_2\eta_2).$$

It is shown that *the probability is expressed by the formula*

$$\kappa_{1\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2) = \kappa(\alpha\beta;\ \xi_1\eta_1)\cdot\bar{A}_{\xi_2\eta_2} + 2^{-\nu}W(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2).$$

The quantity $W(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)$ in the residual term evidently vanishes out unless $A_{\xi_1\eta_1}$ possesses at least a gene in common with $A_{\alpha\beta}$, and its values are given as follows; cf. also a remark stated at the end of I, § 1:

$W(ii;\ ii,\ ii) = 3i^2(1-i)$, $\quad W(ii;\ ii,\ ig) = 3ig(1-2i)$,
$W(ii;\ ii,\ gg) = -3ig^2$, $\quad W(ii;\ ii,\ fg) = -6ifg$,
$W(ii;\ ik,\ ii) = ik(2-3i)$, $\quad W(ii;\ ik,\ ik) = k(i+2k-6ik)$,
$W(ii;\ ik,\ kk) = k^2(1-3k)$, $\quad W(ii;\ ik,\ ig) = 2kg(1-3i)$,
$W(ii;\ ik,\ kg) = kg(1-6k)$, $\quad W(ii;\ ik,\ gg) = -3kg^2$,

1) Cf. a previous paper: IV. Mother-child combinations. **27** (1951), 587-620.

$W(ii;\, ik, fg) = -6kfg;$

$W(ij;\, ii, ii) = \frac{1}{2}i^2(2-3i),$ $W(ij;\, ii, ij) = \frac{1}{2}i(i+2j-6ij),$

$W(ij;\, ii, jj) = \frac{1}{2}ij(1-3j),$ $W(ij;\, ii, ig) = ig(1-3i),$

$W(ij;\, ii, jg) = \frac{1}{2}ig(1-6j),$ $W(ij;\, ii, gg) = -\frac{3}{2}ig^2,$

$W(ij;\, ii, fg) = -3ifg,$

$W(ij;\, ij, ii) = \frac{1}{2}i(2i+j-3i(i+j)),$

$W(ij;\, ij, ij) = \frac{1}{2}(i^2+j^2+4ij-6ij(i+j)),$

$W(ij;\, ij, ig) = \frac{1}{2}g(2i+j-6i(i+j)),$ $W(ij;\, ij, gg) = -\frac{3}{2}g^2(i+j),$

$W(ij;\, ij, fg) = -3fg(i+j),$

$W(ij;\, ik, ii) = \frac{1}{2}ik(1-3i),$ $W(ij;\, ik, ij) = \frac{1}{2}k(i+j-6ij),$

$W(ij;\, jk, jj) = \frac{1}{2}jk(1-3j),$ $W(ij;\, ik, ik) = \frac{1}{2}k(i+k-6ik),$

$W(ij;\, ik, jk) = \frac{1}{2}k(j+k-6jk),$ $W(ij;\, ik, kk) = \frac{1}{2}k^2(1-3k),$

$W(ij;\, ik, ig) = \frac{1}{2}kg(1-6i),$ $W(ij;\, ik, jg) = \frac{1}{2}kg(1-6j),$

$W(ij;\, ik, kg) = \frac{1}{2}kg(1-6k),$ $W(ij;\, jk, gg) = -\frac{3}{2}kg^2,$

$W(ij;\, ik, fg) = -3kfg.$

The proof of the formula is performed by induction by directly verifying an identity

$$\sum W(\alpha\beta;\, \xi_1\eta_1,\, ab)\kappa(ab;\, \xi_2\eta_2) = \sum \kappa(\alpha\beta;\, \xi_1\eta_1,\, ab)Q(ab;\, \xi_2\eta_2)$$
$$= \tfrac{1}{2}W(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2).$$

It is noted that the quantity W satisfies further identities

$$\sum W(\alpha\beta;\, \xi\eta,\, ab) = 0, \qquad \sum W(\alpha\beta;\, ab,\, \xi\eta) = 2Q(\alpha\beta;\, \xi\eta),$$
$$\sum \overline{A}_{ab}W(ab;\, \xi_1\eta_1,\, \xi_2\eta_2) = 2\overline{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\, \xi_2\eta_2).$$

2. Mother-μth descendant-νth descendant combination

The formula for the last generic system with $\mu, \nu > 1$ is expressed in the form

$$\kappa_{\mu\nu}(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2) = \overline{A}_{\xi_1\eta_1}\overline{A}_{\xi_2\eta_2} + 2^{-\mu+1}\overline{A}_{\xi_2\eta_2}Q(\alpha\beta;\, \xi_1\eta_1)$$
$$+2^{-\nu+1}\overline{A}_{\xi_1\eta_1}Q(\alpha\beta;\, \xi_2\eta_2) + 2^{-\lambda}T(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2),$$
$$\lambda = \mu+\nu-1,$$

where the values of T are as follows; cf. a remark stated at the end of I, § 1:

$T(ii;\, ii, ii) = i^2(1-i)(2-i),$ $T(ii;\, ii, ig) = ig(1-2i)(2-i),$

$T(ii;\, ii, gg) = -ig^2(2-i),$ $T(ii;\, ii, fg) = -2ifg(2-i),$

$T(ii;\, ik, ik) = k(2k+i^2-7ik+4i^2k),$ $T(ii;\, ik, kk) = k^2(i-2k+2ik),$

$T(ii;\, ik, ig) = kg(2-7i+4i^2),$ $T(ii;\, ik, kg) = kg(i-4k+4ik),$

$T(ii;\, ik, gg) = -2kg^2(1-i),$ $T(ii;\, ik, fg) = -4kfg(1-i),$

$T(ii;\, kk, kk) = k^3(1+k),$ $T(ii;\, kk, kg) = k^2g(1+2k),$

$T(ii;\, kk, gg) = k^2g^2,$ $T(ii;\, kk, fg) = 2k^2fg,$

$T(ii;\, hk, hk) = hk(h+k+4hk),$ $T(ii;\, hk, kg) = hkg(1+4k),$

$T(ii; hk, fg) = 4hkfg;$

$T(ij; ii, ii) = \frac{1}{2}i^2(1-2i+2i^2),$ $\quad T(ij; ii, ij) = \frac{1}{2}i(1-2i)(i+j-2ij),$

$T(ij; ii, jj) = \frac{1}{2}ij(1-2i-2j+2ij),$ $\quad T(ij; ii, ig) = \frac{1}{2}ig(1-2i)^2,$

$T(ij; ii, jg) = \frac{1}{2}ig(1-2i-4j+4ij),$ $\quad T(ij; ii, gg) = -ig^2(1-i),$

$T(ij; ii, fg) = -2ifg(1-i),$

$T(ij; ij, ij) = \frac{1}{2}(i+j-4ij)(i+j-2ij),$

$T(ij; ij, ig) = \frac{1}{2}g(1-4i)(i+j-2ij),$

$T(ij; ij, gg) = -g^2(i+j-2ij),$ $\quad T(ij; ij, fg) = -2fg(i+j-2ij),$

$T(ij; ik, ik) = \frac{1}{2}k(k+2i^2-6ik+8i^2k),$

$T(ij; ik, jk) = \frac{1}{2}k(k+2ij-4ik-4jk+8ijk),$

$T(ij; ik, kk) = k^2(i-k+2ik),$ $\quad T(ij; ik, ig) = \frac{1}{2}kg(1-4i)(1-2i),$

$T(ij; ik, jg) = \frac{1}{2}kg(1-4i-4j+8ij),$

$T(ij; ik, kg) = kg(i-2k+4ik),$

$T(ij; ik, gg) = -kg^2(1-2i),$ $\quad T(ij; ik, fg) = -2kfg(1-2i).$

The proof of the formula can be performed by induction by means of a recurrence equation

$$\kappa_{\mu\nu}(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2) = \sum \kappa_{\mu-1,\,\nu}(\alpha\beta;\, ab,\, \xi_2\eta_2)\kappa(ab;\, \xi_1\eta_1),$$

together with the identities

$$\begin{aligned}\sum W(\alpha\beta;\, ab,\, \xi_2\eta_2)\kappa(ab;\, \xi_1\eta_1) &= \sum W(\alpha\beta;\, ab,\, \xi_2\eta_2)Q(ab;\, \xi_1\eta_1)\\ &= \tfrac{1}{2}T(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2),\\ \sum T(\alpha\beta;\, ab,\, \xi_2\eta_2)\kappa(ab;\, \xi_1\eta_1) &= \sum T(\alpha\beta;\, ab,\, \xi_2\eta_2)Q(ab;\, \xi_1\eta_1)\\ &= \tfrac{1}{2}T(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2).\end{aligned}$$

It is noted that the quantity T satisfies, besides an evident symmetry relation $T(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2) = T(\alpha\beta;\, \xi_2\eta_2,\, \xi_1\eta_1)$, also an identity

$$\sum T(\alpha\beta;\, ab,\, \xi\eta) = 0, \quad \sum \overline{A}_{ab}T(ab;\, \xi_1\eta_1,\, \xi_2\eta_2) = 2\overline{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\, \xi_2\eta_2).$$

An asymptotic behavior of $\kappa_{\mu\nu}$ as ν (or μ) tends to infinity can be readily deduced. In fact, we get a limit equation

$$\lim_{\nu\to\infty} \kappa_{\mu\nu}(\alpha\beta;\, \xi_1\eta_1,\, \xi_2\eta_2) = \kappa_\mu(\alpha\beta;\, \xi_1\eta_1)\cdot\overline{A}_{\xi_2\eta_2},$$

which remains valid for any μ with $\mu \geqq 1$.