56. *Note on Kodaira-Spencer's Proof of Lefschetz Theorems*

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It has been for a long time expected to have a rigorous proof of the celebrated theorem of Lefschetz on the homomorphism between homology groups of an algebraic variety $V^n (n \geqq 2)$ and those of its generic hyperplane section S. Recently Kodaira and Spencer have succeeded through their deep investigations in proving not only this theorem but also the lemma of Enriques-Severi-Zariski at the same time. By a differential-geometric method Kodaira gained the fundamental lemma on the cohomology groups with coefficients in the stack of germs of holomorphic forms whose coefficients lie in a complex line bundle over V. Standing on this lemma he succeeded also in obtaining the decisive result: the Kähler variety of Hodge's type is equivalent to a projective variety (namely an algebraic variety).

The differential-geometric considerations were necessary indeed for the discovery of the lemma, but not necessary, as we shall show in this note, for the proof itself. But by this simplification we can somewhat improve the lemma and establish the above Lefschetz theorem in the original form.[2)] Further we shall add a remark that the Lefschetz-Hodge's theorem[3)] may be deduced from the fundamental lemma on the same principle.

1. Let V be a compact Kähler variety of complex dim. n, $\mathfrak{U}=\{U_j\}$ a simple covering of V, N its nerve, $\mathfrak{B}$ a complex line bundle defined by $\{f_{jk}\}$ referred to the covering $\mathfrak{U}$, where $f_{jk}(z)$ is a non vanishing holomorphic function in $U_j \frown U_k$ and

$$f_{jk} f_{kj} = 1, \qquad f_{jk} f_{ki} f_{ij} = 1.$$

We shall redefine the characteristic class $c(\mathfrak{B})$ as follows:[4)]

$$c_{ijk} = h_{jk} + h_{ki} + h_{ij}, \qquad h_{jk} = \frac{\sqrt{-1}}{2\pi} \log f_{jk}. \tag{1}$$

Then, if $\mathfrak{B}$ is defined by the divisor $\boldsymbol{D}$ whose local equation is $f_j(z)=0$ and $f_j/f_k = f_{jk}$, considering $\boldsymbol{D}$ as a current we obtain easily a Weil's chain of double cochains[5)] (coélément)

$$\boldsymbol{D} \longleftrightarrow \left(\frac{1}{2\pi\sqrt{-1}} d \log f_j\right) \longleftrightarrow \left(\frac{\sqrt{-1}}{2\pi} \log f_{jk}\right) \longleftrightarrow (c_{ijk}),$$

thus $\boldsymbol{D}$ is cohomologous to $c(\{\boldsymbol{D}\})$.[6)]

Returning to the general case, consider the cochain $(t_{jk}) = (\log|f_{jk}|^2)$ of double degree (0,1), then as $\delta_N(t_{jk})=0$, it is $(t_{jk})=\delta_N \boldsymbol{X}$, $\boldsymbol{X}=(\alpha_j)$, where α_j is real c^∞ in U_j. Putting $e^{-\alpha_j}=a_j$, then $|f_{jk}|^2=a_j/a_k$ and

$$\log f_{jk}+\log \bar{f}_{jk}=\log a_j-\log a_k$$
$$d'\log a_j-d'\log a_k=d'\log f_{jk}=d\log f_{jk}\,.^{7)}$$

Therefore the following chain of double cochains is also the Weil's chain.

$$\frac{1}{2\pi}\chi\longleftrightarrow\left(\frac{\sqrt{-1}}{2\pi}\rho_j\right)\longleftrightarrow\left(\frac{\sqrt{-1}}{2\pi}\log f_{jk}\right)\longleftrightarrow(c_{ijk})=c(\mathfrak{B}),$$

where $\quad \rho_j=-d'\log a_j,\quad \chi=\sqrt{-1}\,d''\rho_j=\sqrt{-1}\,d'd''\log a_j. \qquad (2)$

Thus $c(\mathfrak{B})$ is cohomologous to the real quadratic form $\dfrac{1}{2\pi}\chi$.

2. Let $\Phi_{\mathfrak{B}}$ be the module of differential forms φ whose coefficients lie in $\mathfrak{B}=\{f_{jk}\}$, namely $\varphi=\varphi_j$ in U_j and $\varphi_j=f_{jk}\varphi_k$ in $U_j\frown U_k$. Specially when $\mathfrak{B}=(0)$ (trivial bundle), then Φ_0 is the module of forms on $\boldsymbol{V}$. We define as the inner product (φ,ψ) or $\langle\varphi,\psi\rangle$ of two forms $\varphi,\psi\in\Phi_0$ or $\Phi_{\mathfrak{B}}$ respectively as follows:

$$(\varphi,\psi)=\int_V\varphi\bar{*}\psi,\quad \langle\varphi,\psi\rangle=\int_V\frac{1}{a_j}\varphi_j\bar{*}\psi_j.$$

Then the adjoint operator $\mathfrak{d}$ of d'' with respect to $\langle\ \rangle$ is given by

$$\mathfrak{d}\varphi_i=-*a_jd'\left(\frac{1}{a_j}*\varphi_j\right)=\delta''\varphi_j-*(\rho_j*\varphi_j). \qquad (3)$$

Let $\mathrm{i}(\chi)$ be the adjoint operator of $\mathrm{e}(\chi)$ (the exterior multiplication of real quadratic form χ), then it is easy to see

$$\mathrm{i}(\chi)=(-1)^r\bar{*}\mathrm{e}(\chi)\bar{*}=(-1)^r*\mathrm{e}(\chi)*\quad(\text{for }\Phi\in\Phi^r). \qquad (4)$$

Consider now the involutory mapping $\pi\colon \Phi_{\mathfrak{B}}\to\Phi_{-\mathfrak{B}}$ such that

$$\pi\colon\ \varphi=\{\varphi_j\}\to\varphi'=\{\varphi_j'\},\ \varphi_j'=\frac{1}{a^j}\bar{*}\varphi_j.$$

By easy calculations we get for

$$d''\varphi_j'=(-1)^{r+1}*\left(\frac{1}{a_j}\mathfrak{d}\varphi_j\right),\quad \mathfrak{d}\varphi_j'=-\frac{1}{a_j}*d'(a_j*\varphi_j')=(-1)^r\frac{1}{a_j}\bar{*}d''\varphi_j.$$

If φ is $\square$-harmonic in $\Phi_{\mathfrak{B}}$, then since $d''\varphi=\mathfrak{d}\varphi=0$ the above relations show that $\pi\varphi=\varphi'$ is also $\square$-harmonic in $\Phi_{-\mathfrak{B}}$, and conversely. Therefore, if $H^{p,q}(\mathfrak{B})$ denotes the module of $\square$-harmonic forms of type (p,q) of the bundle $\mathfrak{B}$, it is

$$H^{p,q}(\mathfrak{B})\simeq H^{n-p,\,n-q}(-\mathfrak{B}). \qquad (5)$$

3. Let us now prove $\varphi \in \Phi_{\mathfrak{B}}$ the following:

$$(\mathfrak{d}\delta'+\delta'\mathfrak{d})\varphi_j=\sqrt{-1}\ \mathrm{i}(\chi)\varphi_j. \tag{6}$$

In fact by (3)

$$\mathfrak{d}\delta'\varphi_j=\delta''\delta'\varphi_j-*(\rho_j*\delta'\varphi_j)=\delta''\delta'\varphi_j-(-1)^r*\{\rho_j d''(*\varphi_j)\},$$

$$\delta'\mathfrak{d}\varphi_j=\delta'\delta''\varphi_j-\delta'(*\rho_j*\varphi_j)=\delta'\delta''\varphi_j+(-1)^{r+1}*d''(\rho_j*\varphi_j)$$

$$=\delta'\delta''\varphi_j+(-1)^r*\{\rho_j d''(*\varphi_j)\}-(-1)^r*(d''\rho_j)*\varphi_j.$$

Therefore, by using (4), we get (6):

$$(\mathfrak{d}\delta'+\delta'\mathfrak{d})\varphi_j=-\frac{(-1)^r}{\sqrt{-1}}*\mathrm{e}(\chi)*\varphi_j=\sqrt{-1}\ \mathrm{i}(\chi)\varphi_j.$$

Next also adopting H. Cartan's notation $\mathrm{e}(\Omega)$, $\Omega=\sum\omega_\alpha\bar{\omega}_\alpha$, we have as the fundamental formula for Kähler variety

$$\mathrm{i}(\Omega)d''-d''\,\mathrm{i}(\Omega)=\delta'.$$

If we use Weil's notation $L=\sqrt{-1}\ \mathrm{e}(\Omega)$, then its adjoint is $\Lambda=-\sqrt{-1}\ \mathrm{i}(\Omega)$,[8] and the above formula will be written in the following form, hence δ' is also a linear operator on $\Phi_{\mathfrak{B}}$.

$$d''\Lambda-\Lambda d''=\sqrt{-1}\,\delta'. \tag{7}$$

Let now φ be $\square$-harmonic, then since $\mathfrak{d}\varphi=d''\varphi=0$, it will be

$$\begin{aligned}\langle\Lambda\varphi,\mathfrak{d}\delta'\varphi\rangle&=\langle\Lambda\varphi,(\mathfrak{d}\delta'+\delta'\mathfrak{d})\varphi\rangle\\&=\langle\Lambda\varphi,\sqrt{-1}\ \mathrm{i}(\chi)\varphi\rangle \qquad \text{by } (6)\\&=-\sqrt{-1}\ \langle\Lambda\varphi,\mathrm{i}(\chi)\varphi\rangle\end{aligned}$$

as $\langle\varphi,\alpha\psi\rangle=\bar{\alpha}\langle\varphi,\psi\rangle$. On the other hand, it will be

$$\begin{aligned}\langle\Lambda\varphi,\mathfrak{d}\delta'\varphi\rangle&=\langle d''\Lambda\varphi,\delta'\varphi\rangle\\&=\langle(d''\Lambda-\Lambda d'')\varphi,\delta'\varphi\rangle\\&=\sqrt{-1}\ \langle\delta'\varphi,\delta'\varphi\rangle \qquad \text{by } (7).\end{aligned}$$

Hence we have the important *formula for* $\square$*-harmonic forms*

$$-\langle\Lambda\varphi,\mathrm{i}(\chi)\varphi\rangle=\langle\delta'\varphi,\delta'\varphi\rangle\geqq 0. \tag{8}$$

4. Let $\chi=\sqrt{-1}\sum\chi_{\alpha\beta}dz^\alpha d\bar{z}^\beta$ and assume that the quadratic form $\sum\chi_{\alpha\beta}(t^\alpha,t^\beta)$ [(t^α,t^β): symmetric product] is everywhere positive definite. (Then we say that χ is everywhere positive definite.) In this case we may define the Kähler metric on V taking this form as the fundamental form. Accordingly we can see by (8) that, as we may consider in this case $\Lambda=\mathrm{i}(\chi)$,

$$-\langle\Lambda\varphi,\Lambda\varphi\rangle\geqq 0.$$

Therefore if there exists a $\square$-harmonic form $\varphi\neq 0$ of type (p,q) with $p+q\geqq n+1$, then by a well-known Hodge's theorem on effective form we see $\Lambda\varphi\neq 0$ and $\langle\Lambda\varphi,\Lambda\varphi\rangle>0$. This contradicts with the above inequality, so it must be $\varphi^{p,q}=0$, when $\Box\varphi=0$ and $p+q\geqq n+1$. Thus we get *Kodaira's lemmas*:

Theorem 1. *If the characteristic class* $c(\mathfrak{B})$ *contains an everywhere positive definite quadratic form, namely* $c(\mathfrak{B})>0$, *then*

$$H^{p,q}(\mathfrak{B})\simeq 0 \quad \text{for} \quad p+q\geqq n+1,$$

where n *is the complex dim. of the underlying space* V *of* $\mathfrak{B}$.

By the involutory mapping π: $\Phi_{\mathfrak{B}}\to\Phi_{-\mathfrak{B}}$, we have also

Theorem 1′. *If the characteristic class* $c(\mathfrak{B})$ *contains an everywhere negative definite form, namely* $c(\mathfrak{B})<0$, *then it is*

$$H^{p,q}(\mathfrak{B})\simeq 0 \quad \text{for} \quad p+q\leqq n-1. \tag{9}$$

Further let $\Omega^p(\mathfrak{B})$ be the stack of germs of holomorphic p-forms whose coefficients in $\mathfrak{B}$. Then by the Dolbeault's theorem

$$H^{p,q}(\mathfrak{B})\simeq H^q(\Omega^p(\mathfrak{B}))$$

where $H^q(\Omega^p(\mathfrak{B}))=H^q(V, \Omega^p(\mathfrak{B}))$. Hence

Theorem 1″. *If* $c(\mathfrak{B})$ *contains an everywhere positive or negative definite form, then it is respectively*

$$H^q(\Omega^q(\mathfrak{B}))\simeq 0 \quad \text{for} \quad p+q\geqq n+1 \text{ or } \leqq n-1. \tag{10}$$

5. Let S^{n-1} be a non-singular analytic subvariety of V^n, $\mathfrak{B}_s$ the restriction of the bundle $\mathfrak{B}$ on S, and the sequence

$$0\to\Omega'^p(\mathfrak{B})\xrightarrow{i}\Omega^p(\mathfrak{B})\xrightarrow{r}\Omega^p(\mathfrak{B}_s)\to 0 \tag{11}$$

be exact. And let $\eta\in\Omega'^p(\mathfrak{B})$

$$\eta=\sum_{\alpha_1<\cdots<\alpha_p}\eta_{\alpha_1\cdots\alpha_p}dz^{\alpha_1}\ldots dz^{\alpha_p},$$

where we assume that $z^1=0$ the local equation of S. Then it is clear, that for $p\geqq 1$

$$\eta'=\sum_{1<\alpha_2<\cdots<\alpha_p}(\eta_{1\alpha_2\cdots\alpha_p})_s\,dz^{\alpha_2}\ldots dz^{\alpha_p}$$

belongs to $\Omega^{p-1}\{(\mathfrak{B}-\{s\})_s\}$. If we denote by $\bar{r}$ the mapping $\Omega'^p(\mathfrak{B})\to\Omega^{p-1}\{(\mathfrak{B}-\{s\})_s\}$ such that

$$\bar{r}:\quad \eta\in\Omega'^p(\mathfrak{B})\to\eta'\in\Omega^{p-1}\{(\mathfrak{B}-\{s\})_s\},$$

then we can easily prove that the sequence

$$0\to\Omega^p(\mathfrak{B}-\{s\})\to\Omega'^p(\mathfrak{B})\to\Omega^{p-1}\{(\mathfrak{B}-\{s\})_s\}\to 0 \tag{12}$$

is exact. By taking the sequence of cohomology groups corresponding to (12), we obtain the exact sequence

$$\to H^q(\Omega^p(\mathfrak{B}-\{s\}))\to H^q(\Omega'^p(\mathfrak{B}))\to H^q(\Omega^{p-1}\{(\mathfrak{B}-\{s\})_s\})\to. \tag{13}$$

Now let us assume that S is so ample that $c(\mathfrak{B}-\{s\})$ contains an everywhere negative definite form, then we see by theorem 1″ that

$$H^q(\Omega^p(\mathfrak{B}-\{s\}))\simeq 0 \qquad \text{for} \quad p+q\leqq n-1,$$

$$H^q(\Omega^{p-1}(\mathfrak{B}-\{\boldsymbol{s}\})_s)\simeq 0 \quad \text{for} \quad p+q\leqq n-1.$$

Putting these in (13), we have, if $p\geqq 1$, for $p+q\leqq n-1$

$$H^q(\Omega'^p(\mathfrak{B}))\simeq 0. \tag{14}$$

Moreover it is also the case even when $p=0$, because $\Omega'^0(\mathfrak{B})=\Omega^0(\mathfrak{B}-\{\boldsymbol{s}\})$, $H^q(\Omega^0(\mathfrak{B}-\{\boldsymbol{s}\}))\simeq 0$ for $q\leqq n-1$. On the other hand, taking the sequence of cohomology groups of the sequence (11), we get the exact sequence

$$\rightarrow H^q(\Omega'^p(\mathfrak{B}))\rightarrow H^q(\Omega^p(\mathfrak{B}))\rightarrow H^q(\Omega^p(\mathfrak{B}_s))\rightarrow H^{q+1}(\Omega'^p(\mathfrak{B}))\rightarrow.$$

Theorem 2. *If the divisor $\boldsymbol{S}^{n-1}$ is so ample such that $c(\mathfrak{B}-\{\boldsymbol{s}\})$ contains an everywhere negative definite form, then for $p+q\leqq n-1$ there exists the isomorphism*

$$H^q(\Omega^p(\mathfrak{B}))\rightarrow H^q(\Omega^p(\mathfrak{B}_s))$$

and this is an isomorphism onto or into according as $p+q\leqq n-2$ or $p+q=n-1$.

Consider the special case, where $\boldsymbol{V}$ is a projective variety, $\boldsymbol{S}$ a generic hyperplane section of $\boldsymbol{V}$. Taking $\mathfrak{B}$ as $\{0\}$ ($\{0\}$ the trivial bundle), then clearly $\{0\}-\{\boldsymbol{S}\}$ contains an everywhere negative definite form, so the mapping

$$H^q(\Omega^p(0))\rightarrow H^q(\Omega^p(0_S)) \quad \text{for} \quad p+q\leqq n-1$$

is isomorphic. But we see by the Dolbeault's theorem

$$H^q(\Omega^p(0))\simeq H^{p,q}(\boldsymbol{V},C), \quad H^q(\Omega^p(0_S))\simeq H^{p,q}(\boldsymbol{S},C),$$

where C is complex number field. Hence we have the Lefschetz theorem in the classical form:

Theorem 3. *Let $\boldsymbol{V}$ be an algebraic variety of dim. n without singularities immersed in a projective space, $\boldsymbol{S}$ be a generic hyperplane section of it (consequently $\boldsymbol{S}$ is irreducible and has no singularities), $H(\boldsymbol{V},C)$ the cohomology group of degree r.*

Then $H^r(\boldsymbol{V},C)$ is isomorphic to $H^r(\boldsymbol{S},C)$ if $r\leqq n-2$, and $H^{n-1}(\boldsymbol{V},C)$ is isomorphic to a submodule of $H^{n-1}(\boldsymbol{S},C)$.

6. We can prove directly by theorem 1″ also the following theorem of Kodaira-Spencer, consequently the Lefschetz-Hodge's theorem.

Theorem 4. *Any complex-line-bundle $\mathfrak{B}$ over a compact projective variety $\boldsymbol{V}$ is equivalent to a divisor-class $\boldsymbol{D}$ on $\boldsymbol{V}$.*[9)9a)]

Proof. Let $\mathfrak{B}$ be any given complex-line-bundle over $\boldsymbol{V}$, $\boldsymbol{S}$ a generic hyperplane section of $\boldsymbol{V}$, $\mathfrak{B}_h=\mathfrak{B}+h\{\boldsymbol{S}\}$ for a natural number h and $\boldsymbol{K}$ the canonical divisor of $\boldsymbol{V}$. Consider now the mapping $\bar{r}$ letting correspond to an n-form with coefficients in $\mathfrak{B}_h-\{\boldsymbol{K}\}$ its Poincaré-residue with respect to $\boldsymbol{S}$. Then the sequence of stacks

$$0 \to \Omega^n(\mathfrak{B}_{h-1} - \{\boldsymbol{K}\}) \to \Omega^n(\mathfrak{B}_h - \{\boldsymbol{K}\}) \xrightarrow{\bar{r}} \Omega_S^{n-1}((\mathfrak{B}_{h-1} - \{\boldsymbol{K}\})_S) \to 0$$

is exact, and to this corresponds the exact cohomology sequence:

$$\to H^0(\boldsymbol{V}, \Omega^n(\mathfrak{B}_h - \{\boldsymbol{K}\})) \xrightarrow{\bar{r}^*} H^0(\boldsymbol{S}, \Omega_S^{n-1}((\mathfrak{B}_{h-1} - \{\boldsymbol{K}\})_S)) \\ \to H^1(\boldsymbol{V}, \Omega^n(\mathfrak{B}_{h-1} - \{\boldsymbol{K}\})) \to .$$

If we take h sufficient large, as it can be readily seen, we can assume $c(\mathfrak{B}_{h-1} - \{\boldsymbol{K}\}) > 0$. Hence in the case we have by theorem 1'' that $H^1(\boldsymbol{V}, \Omega^n(\mathfrak{B}_{h-1} - \{\boldsymbol{K}\})) \simeq 0$ and therefore the mapping $\bar{r}^*$ is a homomorphism *onto*.

We shall now prove theorem 4 by induction with respect to dimension n of $\boldsymbol{V}$. Then there exists a divisor $\boldsymbol{D}'$ of $\boldsymbol{S}^{n-1}$ with $(\mathfrak{B}_{h-1} - \{\boldsymbol{K}\})_S = \{\boldsymbol{D}' + (h-2)\boldsymbol{S}' - \boldsymbol{K}'\}$ where $\boldsymbol{S}'$ is a generic hyperplane section of $\boldsymbol{S}^{n-1}$ and $\boldsymbol{K}'$ the canonical divisor of $\boldsymbol{S}$. If we denote by $\bar{\Omega}^q(\boldsymbol{D})$ (or $\bar{\Omega}_S^q(\boldsymbol{D}')$) the stack of germs of meromorphic forms whose divisors are multiples of $-\boldsymbol{D}$ (or $-\boldsymbol{D}'$) on $\boldsymbol{V}$ (or $\boldsymbol{S}$), then we have

$$\begin{aligned} &\dim H^0(\boldsymbol{S}, \Omega_S^{n-1}((\mathfrak{B}_{h-1} - \{\boldsymbol{K}\})_S)) \\ &= \dim H^0(\boldsymbol{S}, \Omega_S^{n-1}(\{\boldsymbol{D}' + (h-2)\boldsymbol{S}' - \boldsymbol{K}'\})) \\ &= \dim H^0(\boldsymbol{S}, \bar{\Omega}_S^{n-1}(\boldsymbol{D}' + (h-2)\boldsymbol{S}' - \boldsymbol{K}')) \\ &= \dim H^0(\boldsymbol{S}, \Omega_S^0(\boldsymbol{D}' + (h-2)\boldsymbol{S}')) \\ &= \dim |\boldsymbol{D}' + (h-2)\boldsymbol{S}'|_S + 1. \end{aligned}$$

Since h is sufficient large we can assume this number is greater than one. Moreover $\bar{r}^*$ is an onto-mapping and therefore

$$\dim H^0(\boldsymbol{V}, \Omega^n(\mathfrak{B}_h - \{\boldsymbol{K}\}) = \dim H^0(\boldsymbol{V}, \Omega^0(\mathfrak{B}_h)) > 1.$$

Hence there exists a non-trivial cross section in $\Omega^0(\mathfrak{B}_h)$, consequently a divisor $\boldsymbol{D}$ of $\boldsymbol{V}$ with $\mathfrak{B} = \{\boldsymbol{D}\}$, q.e.d.

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References

1) The authors are very grateful to Profs. Kodaira and Spencer for their kindness to send us their mimeographed prints before publishing them.

2) Compare Kodaira-Spencer, **6**, Theor. 2.

3) See Kodaira-Spencer, **4**, Theor. 4.

4) This definition is different by sign from that of Kodaira-Spencer, **3**.

5) Weil, **8**.

6) De Rham-Kodaira, **2**, § 28, lemma 3.

7) We use here the same notations as in H. Cartan's seminary, **1**.

8) Since the product is Hermitian

$$(L\varphi, \psi)=(\sqrt{-1}\,\mathrm{e}(\Omega)\varphi, \psi)=(\mathrm{e}(\Omega)\varphi, -\sqrt{-1}\,\psi)=(\varphi, -\sqrt{-1}\,\mathrm{i}(\Omega)\psi)=(\varphi, \Lambda\psi).$$

9) Compare Kodaira-Spencer, **4**, pp. 873–875.

9a) When I finished this paper, Prof. Weil kindly sent me his lecture-prints in which I found this theorem stated in the case when f_{ij} is rational function. Cf. A. Weil, Fiber-spaces in algebraic geometry, p. 26, Univ. of Chicago, Winter, 1952.