# 54. Note on Dirichlet Series. XIII. On the Analogy between Singularities and Order-Directions. II

By Chuji TANAKA

Mathematical Institute, Waseda University, Tokyo

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(1) **Introduction.** Let us put

(1.1) $F(s)=\sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \quad (s=\sigma+it,\ 0\leqq\lambda_1<\lambda_2<\cdots<\lambda_n\to+\infty).$

In the previous note (1), we have proved

**Theorem I** (C. Tanaka). *Let (1.1) be uniformly convergent in the whole plane. If we have*

(1.2)
(i) $\Re(a_n)\geqq 0 \quad (n=1, 2, \ldots)$
(ii) $\lim_{n\to\infty} 1/\lambda_n \log\lambda_n . \log(\cos(\theta_n))=0, \quad \theta_n=\arg(a_n),$

*then $\Im(s)=0$ is the order-direction of (1.1).*

In this note, we shall generalize it as follows:

**Theorem II.** *Let (1.1) be uniformly convergent in the whole plane. Then there exists at least one order-direction in $|\Im(s)|\leqq\pi\delta$, provided that*

(1.3)
(i) $\lim_{n\to\infty} 1/\lambda_n \log\lambda_n . \log|\cos(\theta_n)|=0, \quad \theta_n=\arg(a_n),$
(ii) *the sequence $\{\Re(a_n)\}$ has sign-changes between $\Re(a_{p_\nu})$ and $\Re(a_{1+p_\nu})$ $(\nu=1, 2, \ldots)$, where $\varliminf_{\nu\to\infty}(\lambda_{1+p_\nu}-\lambda_{p_\nu})=g>0$, $\varlimsup_{\nu\to\infty} \nu/r_\nu=\delta\ (\leqq 1/g)$, $r_\nu=1/2.(\lambda_{p_\nu}+\lambda_{1+p_\nu})$.*

**Theorem III.** *Let (1.1) be uniformly convergent in the whole plane. Let the subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ be defined as follows:*

(1.4)
(a) $\varliminf_{k\to\infty}(\lambda_{n_{k+1}}-\lambda_{n_k})>0, \quad \varliminf_{\substack{n,k\to\infty\\ n\neq n_k}}|\lambda_n-\lambda_{n_k}|>0,$
(b) $\varlimsup_{k\to\infty} k/\lambda_{n_k}=\delta.$

*If we have*

(1.5)
(i) $\Re(a_n)\geqq 0$, for $n\,\overline{\in}\,\{n_k\}$,
(ii) $\lim_{n\to\infty\ n\overline{\in}\{n_k\}} 1/\lambda_n \log\lambda_n . \log(\cos(\theta_n))=0,$

*then in $|\Im(s)|\leqq 2\pi\delta$, there exists at least one order-direction of (1.1).*

From theorem III follows immediately

**Corollary.** *Let (1.1) with $\varliminf_{n\to\infty}(\lambda_{n+1}-\lambda_n)>0$ be simply (necessarily absolutely) convergent in the whole plane. If we have $|\arg(a_n)|\leqq\theta<\pi/2$, except for $\{a_{n_k}\}$ such that $\lim_{k\to\infty} k/\lambda_{n_k}=0$, then $\Im(s)=0$ is the order-direction of (1.1).*

(2) **Lemmas.** We need some lemmas.

**Lemma I.** *Let (1.1) be uniformly convergent in the whole plane. Let us put*

(2.1) $$\varphi(z)=\prod_{\nu=1}^{\infty}(1-z^2/r_\nu^2)^k,$$

*where* (i) $k$: *a positive integer,*

(2.2)
(ii) $0<r_1<r_2<r_3 \cdots <r_\nu< \cdots \to +\infty$
(iii) $\underline{\lim}_{\nu\to+\infty}(r_{\nu+1}-r_\nu)=h_1>0, \quad \underline{\lim}_{\nu,n\to\infty}|r_\nu-\lambda_n|=h_2>0,$
(iv) $\overline{\lim}_{\nu\to+\infty}\nu/r_\nu=\delta \quad (\leqq 1/h_1).$

*Then, following propositions hold:*

(2.3)
(a) $f(s)=\sum_{n=1}^{\infty}a_n\varphi(\lambda_n)\exp(-\lambda_n s)$ *is also uniformly everywhere.*
(b) $f(s)$ *has the same order as (1.1).*
(c) *If* $\Im(s)=t$ *is the order-direction of* $f(s)$, *then in* $|\Im(s)-t|\leqq k\pi\delta$, *there exists at least one order-direction of (1.1).*

**Proof.** Two propositions (a) and (b) have been proved in the previous note (2) p. 93, corollary III). Here we shall establish only (c).

By F. Carlson's theorem (6) p. 267), for any given $\varepsilon(>0)$, we have

(2.4) $$|\varphi(z)|<\exp(k(\pi\delta+\varepsilon)|z|) \quad \text{for } |z|>R(\varepsilon).$$

Taking account of (2.4) and H. Cramer-A. Ostrowski's theorem (6) pp. 49-52), we obtain

(2.5) $$f(s)=1/2\pi i.\oint_{|u|=k\pi\delta+\varepsilon}F(s\text{-}u)\Phi(u)\,du \quad (\varepsilon>0),$$

where $\varphi(z)=\sum_{n=0}^{\infty}c_n/n!\,.z^n$, $\Phi(u)=\sum_{n=0}^{\infty}c_n/u^{n+1}$. Hence,

$$M_f(\sigma, t, \varepsilon)=\operatorname*{Max}_{\Re(s)=\sigma,\ |\Im(s)-t|\leqq\varepsilon}|f(s)|$$

$$\leqq \operatorname*{Max}_{\substack{\Re(s)=\sigma,\ |\Im(s)-t|\leqq\varepsilon\\ |u|=k\pi\delta+\varepsilon}}|F(s\text{-}u)|\,.\,1/2\pi.\oint_{|u|=k\pi\delta+\varepsilon}|\Phi(u)||du|=|F(s')|.A\leqq M_F(\sigma', t, k\pi\delta+2\varepsilon).A,$$

where (i) $A=1/2\pi.\oint_{|u|=k\pi\delta+\varepsilon}|\Phi(u)||du|,$

(ii) $\Re(s')=\sigma', \ |\sigma'-\sigma|\leqq k\pi\delta+\varepsilon, \ |\Im(s')-t|\leqq k\pi\delta+2\varepsilon,$

(iii) $M_F(\sigma, t, k\pi\delta+2\varepsilon)=\operatorname*{Max}_{\Re(s)=\sigma',\ |\Im(s)-t|\leqq k\pi\delta+2\varepsilon}|F(s)|.$

Accordingly, by $\lim_{\sigma\to-\infty}\sigma'/\sigma=1$, we get

(2.6) $$\rho_f(\varepsilon)=\overline{\lim_{\sigma\to-\infty}}\,1/(-\sigma).\log^+\log^+M_f(\sigma, t, \varepsilon)$$

$$=\overline{\lim_{\sigma\to-\infty}}\,1/(-\sigma).\log^+\log^+M_F(\sigma, t, k\pi\delta+2\varepsilon)\leqq\rho_F,$$

where $\rho_F$ is the order of $F(s)$. Since $\mathfrak{J}(s)=t$ is the order-direction of $f(s)$, by (2.3) (b)

$$\rho_f(\varepsilon) = \rho_f = \rho_F, \tag{2.7}$$

denoting by $\rho_f$ the order of $f(s)$.

By (2.6) and (2.7),

$$\rho_F = \overline{\lim_{\sigma\to-\infty}}\, 1/(-\sigma).\log^+\log^+ M_F(\sigma, t, k\pi\delta+2\varepsilon).$$

Since $\varepsilon$ is arbitrary, there exists at least one order-direction of $F(s)$ in the strip: $|\mathfrak{J}(s)-t| \leqq k\pi\delta$, which is to be proved.

**Lemma II** (C. Tanaka). *Let us denote by $\sigma_s$ and $\sigma_a$ the simple and absolute convergence-abscissa of (1.1) respectively. Then we have*

$$0 \leqq \begin{Bmatrix} \sigma_s - C \\ \sigma_a - \sigma_s \end{Bmatrix} \leqq \overline{\lim_{x\to\infty}}\, 1/x.\log^+ N(x) \leqq \overline{\lim_{n\to\infty}}\, 1/\lambda_n.\log n$$

*where* $C = \overline{\lim_{n\to\infty}}\, 1/\lambda_n.\log|a_n|$, $\qquad N(x) = \sum_{[x]\leqq\lambda_n<x} 1$ .

This lemma can be proved by entirely similar arguments as in a lemma ( 3) p. 50), so that we omit its proof.

**Lemma III** (C. Tanaka, 5) p. 68 theorem I). *Let (1.1) be uniformly convergent in the whole plane. Then we have*

$$-1/\rho_c \leqq -1/\rho \leqq -1/\rho_c + \overline{\lim_{x\to\infty}}\,(x\log x)^{-1}.\log^+ N(x)$$
$$\leqq -1/\rho_c + \overline{\lim_{n\to\infty}}\,(\lambda_n\log\lambda_n)^{-1}.\log n,$$

*where* $-1/\rho_c = \lim_{n\to\infty}(\lambda_n\log\lambda_n)^{-1}.\log|a_n|$, $\rho$: *the order of (1.1).*

## ( 3 ) Proof of the Theorems

**Proof of theorem II.** Without any loss of generality, we can assume that $\mathfrak{R}(a_1) > 0$. Let us put

$$f(s) = \sum_{n=1}^{\infty} a_n\varphi(\lambda_n)\exp(-\lambda_n s),$$

where, putting $r_\nu = 1/2\,.\,(\lambda_{p_\nu} + \lambda_{1+p_\nu})$, we set $\varphi(z) = \prod_{n=1}^{\infty}(1-z^2/r_\nu^2)$.

Then we have

$$\text{(3.1)}\quad \begin{aligned} &\text{(i)}\ \ \mathfrak{R}(a_n\varphi(\lambda_n)) \geqq 0, \\ &\text{(ii)}\ \lim_{n\to\infty} 1/(\lambda_n\log\lambda_n).\log(\cos(\theta_n')) = 0, \\ &\qquad \theta_n' = \arg(a_n\varphi(\lambda_n)). \end{aligned}$$

The first part of (3.1) is evident. Since $\varphi(\lambda_n)$ is real, on account of (i) we have $\cos(\theta_n') = |\cos(\theta_n)|$, so that from (1.3) (i) immediately follows (ii).

By (3.1), (a) of lemma I and theorem I, $\mathfrak{J}(s)=0$ is the order-direction of $f(s)$. Hence, by (c) of lemma I, in $|\mathfrak{J}(s)| \leqq \pi\delta$ there exists at least one order-direction of $F(s)$.

**Proof of theorem III.** Let (1.1) be of order $\rho$. Let us put

$$F_1(s) = \sum_{k=1}^{\infty} a_{n_k}\exp(-\lambda_{n_k}s), \quad F_2(s) = \sum_{\substack{n=1 \\ n\in\{n_k\}}}^{\infty} a_n\exp(-\lambda_n s).$$

By lemma II, we have

$$\overline{\lim_{k\to\infty}}\, 1/\lambda_{n_k}.\log|a_{n_k}| \leqq \overline{\lim_{n\to\infty}}\, 1/\lambda_n.\log|a_n| \leqq \sigma_s \leqq \sigma_u,$$

where $\sigma_u$: the uniform convergence-abscissa of (1.1). Since we get

(3.2) $$\overline{\lim_{k\to\infty}}\, 1/\lambda_{n_k}.\log|a_{n_k}| = -\infty.$$

By (3.2) and lemma II, $F_1(s)$ is absolutely (a fortiori uniformly) convergent in the whole plane. Hence, $F_2(s)$ is also uniformly convergent everywhere.

Let us denote by $\rho_1$, $\rho_2$ the order of $F_1(s)$ and $F_2(s)$ respectively. By lemma III, $\rho_1$ is determined by

$$\overline{\lim_{k\to\infty}}\, 1/(\lambda_{n_k}\log\lambda_{n_k}).\log|a_{n_k}| = -1/\rho_1.$$

Again by lemma III, we get

$$-1/\rho_1 = \overline{\lim_{k\to\infty}}\, 1/(\lambda_{n_k}\log\lambda_{n_k}).\log|a_{n_k}|$$
$$\leqq \overline{\lim_{n\to\infty}}\, 1/(\lambda_n\log\lambda_n).\log|a_n| = -1/\rho_c \leqq -1/\rho,$$

so that $\rho_1 \leqq \rho$. Hence, taking account of $F(s)=F_1(s)+F_2(s)$, we get easily $\rho_2 \leqq \rho$.

Now we distinguish two cases:

*Case* $\rho_2<\rho$: Then we have evidently $\rho_1=\rho$. By S. Mandelbrojt's theorem ( 4) p. 423, 7) p. 19) in $|\mathfrak{I}(s)|\leqq\pi\delta$ there exists at least one order-direction of $F_1(s)$. Since $\rho_2<\rho$, the order-direction of $F_1(s)$ is also the order-direction of $F(s)$, so that in $|\mathfrak{I}(s)|\leqq\pi\delta$, a fortiori in $|\mathfrak{I}(s)|\leqq 2\pi\delta$, $F(s)$ has at least one order-direction

*Case* $\rho_2=\rho$: Putting $\varphi(z)=\prod_{k=1}^{\infty}(1-z^2/\lambda_{n_k}^2)^2$, we have

$$f(s)=\sum_{n=1}^{\infty} a_n\,\varphi(\lambda_n)\exp(-\lambda_n s) = \sum_{n\bar{\in}\{n_k\}} a_n\varphi(\lambda_n)\exp(-\lambda_n s).$$

Since $\varphi(\lambda_n)>0$, $\arg(a_n\varphi(\lambda_n))=\arg(a_n)=\theta_n$ for $n\,\bar{\in}\,\{n_k\}$, we have evidently

$$\mathfrak{R}(a_n\varphi(\lambda_n)) \geqq 0 \text{ for } n\,\bar{\in}\,\{n_k\},$$
$$\lim_{\substack{n\to\infty\\ n\bar{\in}\{n_k\}}} 1/(\lambda_n\log\lambda_n).\log\{\cos(\arg(a_n\varphi(\lambda_n)))\} = 0.$$

Hence, by theorem I and (a) of lemma I, $\mathfrak{I}(s)=0$ is the order-direction of $f(s)$. Again, by (c) of lemma I, in $|\mathfrak{I}(s)|\leqq 2\pi\delta$ there exists at least one order-direction of $F(s)$.

Thus, in any case, in $|\mathfrak{I}(s)|\leqq 2\pi\delta$, $F(s)$ has at least one order-direction, q.e.d.

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