70. *On the Generation of a Strongly Ergodic Semi-Group of Operators*

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1. Introduction. A fundamental problem of a semi-group of bounded linear operators $T(\xi)$, $0<\xi<\infty$, from a complex Banach space X into itself is to characterize the infinitesimal generator which determines the structure of a semi-group of operators.

Such a problem has been discussed by E. Hille [1][1] and K. Yosida [2] for a semi-group of operators satisfying the following conditions:

(c_1) *$T(\xi)$ is strongly continuous at zero,*

(c_2) $\|T(\xi)\| \leqq 1+\beta\xi$ *for sufficiently small* ξ,

where β is a constant. Later their results were generalized to a semi-group of operators satisfying only the condition (c_1) by R. S. Phillips [3] and the present author [4]. Further this result has been generalized to a strongly measurable semi-group of operators by W. Feller [5].

In this paper we shall deal with the above problem concerning a semi-group of operators which is strongly Abel (or Cesàro) ergodic to the identity at zero.[2] We sketch here our results. The details will appear in the Tôhoku Mathematical Journal.

2. Semi-group of operators strongly Abel ergodic at zero

Let $\{T(\xi);\ 0<\xi<\infty\}$ be a semi-group of operators satisfying the following conditions:

(a) *For each $\xi>0$, $T(\xi)$ is a bounded linear operator from a complex Banach space X into itself and*

$$T(\xi+\eta)=T(\xi)\cdot T(\eta)=T(\eta)\cdot T(\xi).$$

(b) *$T(\xi)$ is strongly measurable in* $(0, \infty)$.

We may further assume the following condition without loss of generality:

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1) Numbers in brackets refer to the references at the end of this paper.

2) After this paper was written up, the author found the abstract of Phillips' paper [6], in which he writes that the necessary and sufficient conditions that a closed linear operator be the c.i.g. (the smallest closed extension of the infinitesimal generator) of a semi-group of operators which is strongly Abel (or Cesàro) ergodic (summable) to the identity at zero are obtained, but the detail is not obvious for the present author.

(c) $||T(\xi)||$ *is bounded at* $\xi=\infty$.

Definition 1. $T(\xi)$ is said to be *strongly Abel ergodic* to the identity at zero if it satisfies the following conditions:

$$\int_0^1 ||T(\xi)||\,d\xi<\infty,$$

$$\lim_{\lambda\to\infty}\lambda\int_0^\infty e^{-\lambda\xi}T(\xi)x\,d\xi=x,\quad x\in X.$$

Definition 2. The set Σ defined by

$$\Sigma=\left\{x\,;\ \lim_{\xi\to 0}\frac{1}{\xi}\int_0^\xi T(\eta)x\,d\eta=x\right\}$$

is said to be $(C,\ 1)$-*continuity set* of $\{T(\xi)\,;\ 0<\xi<\infty\}$.

Definition 3. We define the operator A by

$$Ax=\lim_{h\to 0}\frac{1}{h}[T(h)-I]x$$

whenever the limit on the right hand side exists and belongs to Σ. A is said to be *the infinitesimal generator* of $\{T(\xi)\,;\ 0<\xi<\infty\}$ and the set of elements x, for which Ax exists, will be denoted by $D(A)$.

We obtain first the following

Theorem 1. *Let* $\{T(\xi)\,;\ 0<\xi<\infty\}$ *be a semi-group of operators satisfying the conditions* (a)–(c) *and strongly Abel ergodic to the identity at zero. Then we have*

(i) *for each* λ *such that* $R(\lambda)>0$, *where* $R(\lambda)$ *denotes the real part of* λ, *there exists a bounded linear operator* $R(\lambda\,;\ A)$ *from* X *into* Σ *satisfying the following conditions:*

$$(\lambda-A)R(\lambda\,;\ A)x=x,\quad x\in\Sigma,$$
$$R(\lambda\,;\ A)(\lambda-A)x=x,\quad x\in D(A),$$

(ii) $D(A)$ *is a dense linear subset in* X,

(iii) *there exists a finite positive constant* M *such that*

$$||\lambda R(\lambda\,;\ A)||\leqq M,\quad \lambda\geqq 1,$$

(iv) *there exists a non-negative function* $f(\xi,\ x)$ *defined on the product space* $<0,\ \infty>\times X$ *satisfying the properties* (a′)–(d′):

(a′) *for each* $x\in X$, $f(\xi,\ x)$ *is a measurable function of* ξ,

(b′) $f(\xi)=\sup_{x\in X}\dfrac{f(\xi,x)}{||x||}$ *is integrable on any finite interval* $[0,\ \epsilon]$ *and bounded measurable on any infinite interval* $[\epsilon,\ \infty]$, $\epsilon>0$,

(c′) $\sup_{x\in X}\dfrac{f(\xi,\ R(1\,;\ A)x)}{||x||}$ *is bounded on* $(0,\ \infty)$,

(d′) *for all* $x\in X$, *we have*

$$||R^{(k)}(\lambda;\ A)x||\leqq(-1)^k F^{(k)}(\lambda,\ x),\ k=1,2,\ldots,$$

where $F(\lambda, x)$ *is defined by*

$$F(\lambda, x)=\int_0^{\infty} e^{-\lambda\xi} f(\xi, x)\, d\xi, \quad \lambda>0,$$

and $R^{(k)}(\lambda; A)$, $F^{(k)}(\lambda, x)$ *denote the k-th derivative of* $R(\lambda; A)$, $F(\lambda, x)$ *respectively,*

(v) *if we define the new norm by*

$$N(x)=\sup_{\xi>0}\left\|\frac{1}{\xi}\int_0^{\xi} T(\eta)x\, d\eta\right\|, \quad x \in \Sigma,$$

then Σ *is a Banach space with* $N(x)$*-norm and* $D(A)$ *is dense in* Σ *with* $N(x)$*-norm, and furthermore*

$$N(x)=\sup_{k\geq 1,\, \lambda>0}\left\|\frac{1}{k}\sum_{i=1}^{k}[\lambda R(\lambda; A)]^i x\right\|, \quad x \in \Sigma.$$

Next we state the converse of Theorem 1.

Theorem 2. *Let* Σ *be a linear subset in* X *and* A *be a linear operator on* Σ *into itself satisfying the conditions* (i)–(iv), *where* $D(A)$ *denotes the domain of* A. *We assume further that* $N(x)$ *defined by*

$$N(x)=\sup_{k\geq 1,\, \lambda>0}\left\|\frac{1}{k}\sum_{i=1}^{k}[\lambda R(\lambda; A)]^i x\right\|, \quad x \in \Sigma,$$

is finitely valued and Σ *is a Banach space with* $N(x)$*-norm and that* $D(A)$ *is dense in* Σ *with* $N(x)$*-norm.*

Then there exists a semi-group of operators $\{T(\xi); 0<\xi<\infty\}$ *such that* $T(\xi)$ *satisfies the conditions* (a)–(c), *is strongly Abel ergodic to the identity at zero and* A *is its infinitesimal generator, and* Σ *is the* $(C, 1)$*-continuity set of* $\{T(\xi); 0<\xi<\infty\}$ *and*

$$N(x)=\sup_{\xi>0}\left\|\frac{1}{\xi}\int_0^{\xi} T(\eta)x\, d\eta\right\|, \quad x \in \Sigma.$$

Theorem 2 is proved by the idea due to K. Yosida and W. Feller.

3. Semi-group of operators strongly (C, 1)-ergodic at zero

Definition 4. $T(\xi)$ is said to be *strongly* $(C, 1)$*-ergodic* to the identity at zero if it satisfies the followings:

$$\int_0^1 \| T(\xi) \|\, d\xi < \infty,$$

$$\lim_{\xi\to 0}\frac{1}{\xi}\int_0^{\xi} T(\eta)x\, d\eta = x, \quad x \in X.$$

In this case the $(C, 1)$-continuity set of $\{T(\xi); 0<\xi<\infty\}$ coincides with the whole space X, so that our definition of the infinitesimal generator (cf. Def. 3) becomes the ordinary one, further $N(x)$-norm defined by

$$N(x)=\sup_{\xi>0}\left\|\frac{1}{\xi}\int_0^{\xi} T(\eta)x\, d\eta\right\|, \quad x \in X$$

is equivalent to the original norm.

In fact, by the conditions (a)–(c) and

$$\lim_{\xi\to 0}\frac{1}{\xi}\int_0^{\xi} T(\eta)x\,d\eta=x, \quad x\in X,$$

there exists a finite positive constant M such that

$$\sup_{\xi>0}\left\|\frac{1}{\xi}\int_0^{\xi} T(\eta)x\,d\eta\right\|\leqq M\|x\|$$

for all $x\in X$, while

$$\|x\|\leqq\sup_{\xi>0}\left\|\frac{1}{\xi}\int_0^{\xi} T(\eta)\,x\,d\eta\right\|,$$

so that we have

$$(*)\qquad \|x\|\leqq\sup_{\xi>0}\left\|\frac{1}{\xi}\int_0^{\xi} T(\eta)x\,d\eta\right\|=N(x)\leqq M\|x\|.$$

We denote the infinitesimal generator of $\{T(\xi);\,0<\xi<\infty\}$ by A and the domain of A by $D(A)$.

Theorem 3. *Let* $\{T(\xi);\,0<\xi<\infty\}$ *be a semi-group of operators satisfying the assumptions* (a)–(c) *and strongly* $(C,1)$*-ergodic to the identity at zero. Then*

(i′) *A is a closed linear operator and its spectrum is located in $R(\lambda)\leqq 0$,*

(ii′) *$D(A)$ is a dense linear subset in X,*

(iii′) *there exists a finite positive constant M such that*

$$\sup_{k\geqq 1,\,\lambda>0}\left\|\frac{1}{k}\sum_{i=1}^{k}[\lambda R(\lambda;A)]^i x\right\|\leqq M\|x\|$$

for all $x\in X$,

(iv′) *the condition* (iv) *is satisfied.*

Proof. Since $\frac{d}{d\xi}T(\xi)x=T(\xi)Ax$ for $x\in D(A)$, we have

$$\frac{1}{\xi}[T(\xi)-I]x=\frac{1}{\xi}\int_0^{\xi} T(\eta)Ax\,d\eta, \quad x\in D(A).$$

Suppose that $\{x_n\}$ is a sequence in $D(A)$ and that $x_n\to x$, $Ax_n\to y$. The above formula holds for $x=x_n$ so that

$$\frac{1}{\xi}[T(\xi)x_n-x_n]=\frac{1}{\xi}\int_0^{\xi} T(\eta)Ax_n\,d\eta.$$

Letting $n\to\infty$, one obtains

$$\frac{1}{\xi}[T(\xi)x-x]=\frac{1}{\xi}\int_0^{\xi} T(\eta)y\,d\eta.$$

Because of the strong $(C,1)$-ergodicity, the right hand side tends to y when $\xi\to 0$. Hence Ax exists and equals to y, so that A is a closed linear operator.

We note next that the $(C, 1)$-ergodicity implies the Abel ergodicity and $\Sigma=X$, and then we have the conclusions (i′)–(iv′) from Theorem 1 and the inequality $(*)$.

The converse of this theorem is stated as follows.

Theorem 4. *Let A be a closed linear operator on X into itself satisfying the conditions* (i′)–(iv′). *Then there exists a semi-group of operators $\{T(\xi);\ 0<\xi<\infty\}$ such that $T(\xi)$ satisfies the conditions* (a)–(c), *is strongly $(C, 1)$-ergodic to the identity at zero and that A is its infinitesimal generator.*

Proof. If we denote the resolvent of A by $R(\lambda; A)$ for each λ such that $R(\lambda)>0$, we have the first resolvent equation by the assumption (i'). In virtue of the assumption (iii') we have $\|\lambda R(\lambda; A)\|\leqq M$, so that we obtain

$$\lim_{\lambda\to\infty}\|\lambda R(\lambda; A)x-x\|=0$$

for all $x\in X$. From this we have

$$\|x\|\leqq\sup_{k\geq 1,\ \lambda>0}\left\|\frac{1}{k}\sum_{i=1}^{k}[\lambda R(\lambda; A)]^i x\right\|\leqq M\|x\|$$

for all $x\in X$, and hence if we take the whole space X as Σ, our assumptions imply the assumptions of Theorem 2. Thus there exists a semi-group of operators $\{T(\xi);\ 0<\xi<\infty\}$ such that $T(\xi)$ satisfies the conditions (a)–(c) and is strongly Abel ergodic to the identity at zero, and such that the whole space X is the $(C, 1)$-continuity set of $\{T(\xi);\ 0<\xi<\infty\}$ and A is its infinitesimal generator. Hence it follows that $T(\xi)$ is strongly $(C, 1)$-ergodic to the identity at zero. This completes the proof.

From Theorems 3 and 4 we get the following corollary.

Corollary. *A necessary and sufficient condition that a closed linear operator A becomes the infinitesimal generator of a semi-group of operators $\{T(\xi);\ 0<\xi<\infty\}$ satisfying the conditions* (a), (c) *and* (c_1), *is that the conditions* (i′) *and* (ii′) *are satisfied, and that there exists a finite positive constant M such that*

$$\|[\lambda R(\lambda; A)]^k\|\leqq M,\quad \lambda>0,\quad k=1, 2, \dots .$$

To prove the necessity it is sufficient to note that there exists a finite positive constant M such that $\|T(\xi)\|\leqq M$ for $0<\xi<\infty$ and the strong continuity implies the strong $(C, 1)$-ergodicity.

If we put $f(\xi, x)=M\|x\|$, then the conditions imply the assumptions of Theorem 4, while $\|T(\xi)\|\leqq M$ follows from the condition $\|[\lambda R(\lambda; A)]^k\|\leqq M$. Thus the sufficiency of the conditions is established by use of Theorem 4.

References

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