90. On the Class S_λ

By Osamu ISHIKAWA
Mathematical Institute, Tokyo Metropolitan University

(Comm. by Z. SUETUNA, M.J.A., June 12, 1954)

§1. **Introduction.** The function $f(x, y)$, which is defined and non-negative in a planer region D, is called to belong to the class S_λ, if the following conditions are satisfied:

(a) $f(x, y)$ is twice continuously differentiable in D and for any point (x, y) in D

$$\lim_{r\to 0}\frac{8}{r^2}\left[\left\{\frac{1}{2\pi}\int_0^{2\pi} f(x+r\cos\theta,\ y+r\sin\theta)\,d\theta\right\}^\lambda - \frac{1}{\pi r^2}\iint_{\xi^2+\eta^2\leqq r^2} f^\lambda(x+\xi,\ y+\eta)\,d\xi\,d\eta\right]\geqq 0\,(\lambda>0),$$

or more generally

(b) $f(x, y)$ is the limit of a decreasing sequence $\{f_n(x, y)\}$ each of which satisfies the condition (a).

In particular, when $\lambda=2$, S_λ is identical with the class of non-negative subharmonic functions, and when $\lambda=2$, S_λ becomes the P.L. class.

The following two theorems for subharmonic function are well-known. The former was proved by T. Radó [1], and the latter by E. F. Beckenbach [2].

Theorem A. *If $f(x, y)$ is non-negative in D and if for any pair of two real constants α and β the function $\{(x-\alpha)^2+(y-\beta)^2\}f(x, y)$ is subharmonic in D, then $f(x, y)$ is a function of the P.L. class in D.*

Theorem B. *If $f(x, y)$ is non-negative in D and if for any pair of two real constants α and β the function $e^{\alpha x+\beta y}f(x, y)$ is subharmonic in D, then $f(x, y)$ is a function of the P.L. class in D.*

In this paper we shall generalize these theorems to the S_λ class.

§2. We require a lemma which plays the fundamental rôle in §3.

Lemma. *Let $f(x, y)$ be non-negative, and twice continuously differentiable in D.*

In order that $f(x, y)$ belongs to the class S_λ, it is necessary and sufficient that

$$f\Delta f-(\lambda-1)(f_x^2+f_y^2)\geqq 0 \text{ in } D.$$

Proof. Let (x, y) be any point in D. Without loss of generality we can assume that $f(x, y)>0$ in D. Since $f(x, y)$ is twice continuously differentiable in D, we have for sufficiently small $r>0$,

$$f(x+r\cos\theta,\ y+r\sin\theta)=f(x,y)+f_x(x,y)r\cos\theta+f_y(x,y)r\sin\theta$$
$$+\frac{1}{2!}\{f_{xx}(x,y)r^2\cos^2\theta+2f_{xy}(x,y)r^2\sin\theta\cos\theta+f_{yy}(x,y)r^2\sin\theta\}+0(r^2).$$

Therefore

$$\frac{1}{2\pi}\int_0^{2\pi}f(x+r\cos\theta,\ y+r\sin\theta)\,d\theta=f(x,y)+\frac{1}{4}r^2\{f_{xx}(x,y)+f_{yy}(x,y)\}+0(r^2).$$

Since $f(x, y) > 0$ in D, we obtain

$$(1)\qquad \left\{\frac{1}{2\pi}\int_0^{2\pi}f(x+r\cos\theta,\ y+r\sin\theta)\,d\theta\right\}^\lambda$$
$$=f^\lambda(x,y)+\frac{\lambda r^2}{4}f^{\lambda-1}(x,y)\{f_{xx}(x,y)+f_{yy}(x,y)\}+0(r^2).$$

However,

$$f^\lambda(x+\rho\cos\theta,\ y+\rho\sin\theta)=f^\lambda(x,y)+\lambda\rho\cos\theta\, f^{\lambda-1}(x,y)f_x(x,y)$$
$$+\lambda\rho\sin\theta\, f^{-1}(x,y)f_y(x,y)+\frac{1}{2!}\{\rho^2\lambda(\lambda-1)\cos^2\theta\, f^{\lambda-2}(x,y)f_x^2(x,y)$$
$$+\rho^2\lambda(\lambda-1)\sin^2\theta\, f^{\lambda-2}(x,y)\, f_y^2(x,y)+2\rho^2\cos\theta\sin\theta$$
$$[\lambda(\lambda-1)f_x(x,y)f_y(x,y)f^{\lambda-2}(x,y)+\lambda f^{\lambda-1}(x,y)f_{xy}(x,y)]$$
$$+\rho^2\lambda\cos^2\theta\, f^{\lambda-1}(x,y)f_{xx}(x,y)+\rho^2\lambda\sin^2\theta\, f^{\lambda-1}(x,y)f_{yy}(x,y)\}+0(\rho^2).$$

For sufficiently small r, and for any $\rho\,(0<\rho\leqq r)$, we have

$$(2)\qquad \frac{1}{\pi r^2}\int_0^{2\pi}\int_0^r f^\lambda(x+\rho\cos\theta,\ y+\rho\sin\theta)\rho\,d\rho\,d\theta$$
$$=f^\lambda(x,y)+\frac{r^2}{8}\lambda(\lambda-1)f^{\lambda-2}(x,y)[f_x^2(x,y)+f_y^2(x,y)]$$
$$+\frac{r^2}{8}\lambda f^{\lambda-1}(x,y)[f_{xx}(x,y)+f_{yy}(x,y)]+0(r^2).$$

By (1) and (2), we see that

$$(3)\qquad \frac{8}{r^2}\left[\left(\frac{1}{2\pi}\int_0^{2\pi}f(x+r\cos\theta,\ y+r\sin\theta)\,d\theta\right)^\lambda\right.$$
$$\left.-\frac{1}{\pi r^2}\int_0^{2\pi}\int_0^r f^\lambda(x+\rho\cos\theta,\ y+\rho\sin\theta)\,\rho\,d\rho\,d\theta\right]$$
$$=\lambda f^{\lambda-1}(x,y)\{f(x,y)[f_{xx}(x,y)+f_{yy}(x,y)]$$
$$-(\lambda-1)[f_x^2(x,y)+f_y^2(x,y)]\}+0(1).$$

Thus we have

$$\lim_{r\to 0}\frac{8}{r^2}\left[\left\{\frac{1}{2\pi}\int_0^{2\pi}f(x+r\cos\theta,\ y+r\sin\theta)\,d\theta\right\}^\lambda\right.$$
$$\left.-\frac{1}{\pi r^2}\int_0^{2\pi}\int_0^r f^\lambda(x+\rho\cos\theta,\ y+\rho\sin\theta)\,\rho\,d\rho\,d\theta\right]$$
$$=\lambda f^{\lambda-1}(x,y)\{f(x,y)[f_{xx}(x,y)+f_{yy}(x,y)]-(\lambda-1)[f_x^2(x,y)+f_y^2(x,y)]\}.$$

Hence the lemma is proved.

Remark. The class S_λ decreases as λ increases.

§3. We shall now generalize Theorem A.

Theorem 1. *If $f(x, y)$ is non-negative in D and for any pair of real constants α and β the function $\{(x-\alpha)^2+(y-\beta)^2\}f(x, y)$ belongs to the class S_λ in D, where $0<\lambda<2$, then $f(x, y)$ belongs to the P.L. class in D.*

Proof. Firstly we assume that $f(x, y)$ is twice continuously differentiable in D. Without loss of generality we can assume that $f(x, y)>0$ in D.

Let us put

$$(4) \qquad F=[(x-\alpha)^2+(y-\beta)^2]f(x, y)=p(x, y)f(x, y).$$

Therefore

$$F_x=2(x-\alpha)f+pf_x, \qquad F_{xx}=2f+4(x-\alpha)f_x+pf_{xx} \qquad \text{and so on.}$$

Then we have

$$(5) \qquad F\Delta F-(\lambda-1)(F_x^2+F_y^2)=pf(p\Delta f+4(x-\alpha)f_x+4(y-\beta)f_y+4f) \\ -(\lambda-1)[p^2(f_x^2+f_y^2)+4f^2p+4f(x-\alpha)pf_x+4f(y-\beta)pf_y] \\ =pR,$$

say. Hence

$$(6) \qquad R=pf\Delta f+4(x-\alpha)(2-\lambda)ff_x+4(y-\beta)(2-\lambda)ff_y \\ -(\lambda-1)p(f_x^2+f_y^2)+4(2-\lambda)f^2.$$

Since $F=pf$ is a function of the class S_λ, we get, by Lemma,

$$(7) \qquad F\Delta F-(\lambda-1)(F_x^2+F_y^2)\geqq 0.$$

By our assumptions, (5), (6) and (7), we have for any pair of real numbers α and β,

$$(8) \qquad R\geqq 0.$$

Therefore $\quad [f\Delta f-(\lambda-1)(f_x^2+f_y^2)]f^2(2-\lambda)\geqq(2-\lambda)^2(f_x^2+f_y^2)f^2.$

Since $2>\lambda>0$, we get

$$f\Delta f-(\lambda-1)(f_x^2+f_y^2)\geqq(2-\lambda)(f_x^2+f_y^2).$$

Thus

$$(9) \qquad f\Delta f-(f_x^2+f_y^2)\geqq 0.$$

That is to say, f belongs to the class S_2, i.e. to the P.L. class. Hence the theorem is proved under the assumption made at the beginning of the proof.

We shall now prove that the assumption made at the beginning of the proof may be dropped. By the definition of the class S_λ, we can find a sequence of functions $\{f_n(x, y)\}$ satisfying the conditions (a) such that $\{(f_x-\alpha)^2+(y-\beta)^2\}f_n(x, y)$ is a function of the class S_λ in D for any pair of real numbers α and β. From the above proved fact, we see that each $f_n(x, y)$ belongs to the P.L. class, and hence $f(x, y)$ is also. Therefore Theorem 1 is completely proved.

We shall now give a generalization of Theorem B.

Theorem 2. *If $f(x, y)$ is non-negative in D and for any pair of real constants α and β the function $e^{\alpha x+\beta y}\times f(x, y)$ belongs to the class S_λ in D, where $0<\lambda<2$, then $f(x, y)$ belongs to the P.L. class in D.*

Proof. As in Theorem 1, we shall first prove the theorem in the case that $f(x, y)$ is twice continuously differentiable in D. Without loss of generality we can assume that $f(x, y)>0$ in D.
If we put

$$F=e^{\alpha x+\beta y}\times f(x, y)=e^{p(x, y)} f(x, y), \tag{10}$$

then we have

$$F_x=e^p\alpha f+e^p f_x, \qquad F_{xx}=e^p\alpha^2 f+2e^p\alpha f_x+e^p f_{xx},$$

and so on, and then

$$\begin{aligned} F\Delta F-(\lambda-1)(F_x^2+F_y^2)&=e^p f[e^p f(\alpha^2+\beta^2)+2e^p(\alpha f_x+\beta f_y)+e^p\Delta f)] \\ &\quad -(\lambda-1)[e^{2p}\alpha^2 f^2+e^{2p}\beta^2 f^2+e^{2p}(f_x^2+f_y^2)+2e^{2p}(\alpha f f_x+\beta f f_y)] \\ &=e^{2p}R, \end{aligned} \tag{11}$$

say. Thus we get

$$R=(2-\lambda)f^2(\alpha^2+\beta^2)-2(2-\lambda)(\alpha f f_x+\beta f f_y)+f\Delta f-(\lambda-1)(f_x^2+f_y^2). \tag{12}$$

Since $F=e^p f$ belongs to the class S_λ, we get, by Lemma,

$$F\Delta F-(\lambda-1)(F_x^2+F_y^2)\geqq 0. \tag{13}$$

By (11), (12), (13) and the assumptions of the theorem, we get for any pair of real numbers α and β,

$$R\geqq 0. \tag{14}$$

Therefore

$$(2-\lambda)f^2(f\Delta f-(\lambda-1)(f_x^2+f_y^2))\geqq(2-\lambda)^2 f^2(f_x^2+f_y^2).$$

Since $(2-\lambda)f>0$, we get

$$f\Delta f-(f_x^2+f_y^2)\geqq 0.$$

Hence f belongs to the P.L. class.

Proof of the general case may be carried out by the similar method as in the proof of Theorem 1. Theorem 2 is now completely proved.

References

[1] T. Radó: Remarques sur les fonctions subharmoniques, C. R., **186**, 346–348 (1928).

[2] E. F. Beckenbach: Functions having subharmonic logarithms, Duke Math. Jour., **8**, 393–400 (1941).