147. On Torus Cohomotopy Groups

By Kiyoshi AOKI

(Comm. by Z. SUETUNA, M.J.A., Oct. 12, 1954)

1. The main object of this note is an application of my theorem in the note [1]. Torus homotopy groups are defined by Fox [2], [3]; but in this note I have adopted another meaning of the torus, and the methods of the paper are strongly influenced by Spanier's paper [4].

2. In this section and the followings, I will use the definitions and lemmas of my note [1], which we refer to as [D].

Lemma 2.1. Let (X, A) be a compact pair with dim $(X-A) < 4n-1$. If $\alpha, \beta, \alpha', \beta' : (X, A) \rightarrow (T^{2n}, q)$ with $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$ and if $g : (X, A) \rightarrow (T^{2n} \vee T^{2n}, (q, q))$ is a normalization of $\alpha \times \beta$ and g': $(X, A) \rightarrow (T^{2n} \vee T^{2n}, (q, q))$ is a normalization of $\alpha' \times \beta'$, then $\Omega g \simeq \Omega g'$.

Proof. Since $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$, $\alpha \times \beta \simeq \alpha' \times \beta'$. Then $g \simeq \alpha \times \beta \simeq \alpha' \times \beta' \simeq g'$. Hence, there is a map

$$F : (X \times I, A \times I) \rightarrow (T^{2n} \times T^{2n}, (q, q))$$

such that

$$\begin{matrix} F(x, 0) = g(x) \\ F(x, 1) = g'(x) \end{matrix} \quad \text{for all } x \in X.$$

Then $(X \times 0) \cup (X \times 1) \subset F^{-1}(T^{2n} \vee T^{2n})$, by [$D$], Lemma 2.3, dim $M < 4n$ for any closed $M \subset X \times I - A \times I$. Hence by [D] Theorem 3.5, a normalization G of F exists such that $G(x, t) = F(x, t)$ for $(x, t) \in F^{-1}(T^{2n} \vee T^{2n})$. That is, there is a map

$$G : (X \times I, A \times I) \rightarrow (T^{2n} \vee T^{2n}, (q, q))$$

such that

$$\begin{matrix} G(x, 0) = F(x, 0) = g(x) \\ G(x, 1) = F(x, 1) = g'(x) \end{matrix} \quad \text{for all } x \in X.$$

Then $\Omega G : (X \times I, A \times I) \rightarrow (T^{2n}, q)$ is a homotopy between Ωg and $\Omega g'$.

Theorem 2.2. If (X, A) is a compact pair with dim $(X-A) < 2n-1$, the homotopy classes $\{\alpha\}$ of maps α of (X, A) into (T^{2n}, q) form an abelian group with the law of composition $\{\alpha\} + \{\beta\} = \{\alpha < f > \beta\}$, where f is an arbitrary normalization of $\alpha \times \beta$.

Proof. [D] Theorem 3.5 implies that a normalization f of $\alpha \times \beta$ exists. Lemma 2.1 of the present note shows that $\{\alpha < f > \beta\}$ does not depend on the choice of $\alpha \in \{\alpha\}$, $\beta \in \{\beta\}$ nor upon the normalization f involved. Therefore, the class $\{\alpha < f > \beta\}$ is uniquely determined by the class $\{\alpha\}$ and $\{\beta\}$.

(a) Commutativity. Let F be a normalizing homotopy for $\alpha\times\beta$. Let $w:(T^{2n}\times T^{2n},(q,q))\to(T^{2n}\times T^{2n},(q,q))$ be defined by $w(y,y')=(y',y)$. Then wF is a normalizing homotopy for $\beta\times\alpha$. Hence, if f is the normalization of $\beta\times\alpha$ determined by F, wf is the normalization of $\beta\times\alpha$ determined by wF. Since $\Omega w(z)=\Omega(z)$ for $z\in T^{2n}\vee T^{2n}$, we see that

$$\alpha<f>\beta=\Omega f=\Omega wf=\beta<wf>\alpha.$$

Therefore,

$$\{\alpha\}+\{\beta\}=\{\beta\}+\{\alpha\}.$$

(b) Associativity. Let $\alpha,\beta,\gamma:(X,A)\to(T^{2n},q)$ be any three maps. Then $\alpha\times\beta\times\gamma:(X,A)\to(T^{2n}\times T^{2n}\times T^{2n},(q,q,q))$. Subdivide $T^{2n}\times T^{2n}\times T^{2n}$ simplicially so that (q,q,q) is a vertex and $(\bar{q},\bar{q},\bar{q})\cup T^{2n}\times T^{2n}\times\bar{p}\times p\cup T^{2n}\times T^{2n}\times p\times\bar{p}\cup T^{2n}\times\bar{p}\times p\times T^{2n}\cup T^{2n}\times p\times\bar{p}\times T^{2n}\cup\bar{p}\times p\times T^{2n}\times T^{2n}\cup p\times\bar{p}\times T^{2n}\times T^{2n}$ is in the interior of $T^{2n}\times T^{2n}\times\sigma^{2n}\cup T^{2n}\times\sigma^{2n}\times T^{2n}\cup\sigma^{2n}\times T^{2n}\times T^{2n}$, where its closure $T^{2n}\times T^{2n}\times\bar{\sigma}^{2n}\cup T^{2n}\times\bar{\sigma}^{2n}\times T^{2n}\cup T^{2n}\times T^{2n}\times T^{2n}$ does not meet (q,q,q). Since $\dim(X-A)<2n-1<6n$, [D] Lemma 3.4 shows that there is a map

$$g:(X,A)\to(T^{2n}\times T^{2n}\times T^{2n},(q,q,q))$$

such that $\alpha\times\beta\times\gamma\simeq g$ and

$$g(X)\subset T^{2n}\times T^{2n}\times T^{2n}-T^{2n}\times T^{2n}\times\sigma\cup T^{2n}\times\sigma\times T^{2n}\cup\sigma\times T^{2n}\times T^{2n}.$$

[D] Theorem 3.2 shows that there is a map

$$g':(X,A)\to(T^{2n}\times T^{2n}\times T^{2n},(q,q,q))$$

such that $g'\simeq g$ and $g'(X)\subset(T^{2n}\times T^{2n}\times q)\cup(T^{2n}\times q\times T^{2n})\cup(q\times T^{2n}\times T^{2n})$.

Let

$$M_1=g'^{-1}(q\times T^{2n}\times T^{2n}),\ M_2=g'^{-1}(T^{2n}\times q\times T^{2n}),$$
$$M_3=g'^{-1}(T^{2n}\times T^{2n}\times q).$$

Then M_i is a closed subset of X, so $\dim(M_i-A)<4n$. Let $g'_i=g'\mid M_i$ $(i=1,2,3)$. By [D] Theorem 3.5, there exists a normalization h_i of g'_i such that $h_i\simeq g'_i$ rel $g_i'^{-1}(T^{2n}\vee T^{2n}\vee T^{2n})$ where $T^{2n}\vee T^{2n}\vee T^{2n}=(T^{2n}\times q\times q)\cup(q\times T^{2n}\times q)\cup(q\times q\times T^{2n})$.

Note that $g_i'^{-1}(T^{2n}\vee T^{2n}\vee T^{2n})=M_i\cap(M_j\cup M_k)$ for $(i,j,k)=(1,2,3)$.

Define

$$h:(X,A)\to(T^{2n}\times T^{2n}\times T^{2n},(q,q,q))$$

by

$$h(x)=h_i(x)\quad\text{if }x\in M_i\qquad(i=1,2,3).$$

Then $h(X)\subset T^{2n}\vee T^{2n}\vee T^{2n}$ and $h\mid M_i\simeq g'\mid M_i$ rel $M_i\cap(M_j\cup M_k)$.

Hence, these homotopies can be put together to give a homotopy $h\simeq g'$. Therefore, $h\simeq\alpha\times\beta\times\gamma$.

Let $\pi_{ij}:(T^{2n}\times T^{2n}\times T^{2n},(q,q,q))\to(T^{2n}\times T^{2n},(q,q))$ be defined by $\pi_{ij}(y_1,y_2,y_3)=(y_i,y_j)$. Then $\pi_{12}(\alpha\times\beta\times\gamma)=\alpha\times\beta$, $\pi_{12}h\simeq\alpha\times\beta$, and $\pi_{12}h(X)$

$\subset T^{2n} \vee T^{2n}$. Hence $\pi_{12}h$ is a normalization of $\alpha \times \beta$, so $\alpha < \pi_{12}h > \beta$ determines the sum of α and β.
Let $\pi_i : (T^{2n} \times T^{2n} \times T^{2n}, (q, q, q)) \to (T^{2n}, q)$ be defined by

$$\pi_i(y_1, y_2, y_3) = y_i.$$

Then

$\pi_3 h \simeq \pi_3(\alpha \times \beta \times \gamma) = \gamma$, so $(\alpha < \pi_{12}h > \beta) \simeq \Omega\pi_{12}h$. Let $\Omega_{12} : (T^{2n} \vee T^{2n} \vee T^{2n}, (q, q, q)) \to (T^{2n} \vee T^{2n}, (q, q))$ be defined by

$$\Omega_{12}(y_1, y_2, y_3) = (\Omega(y_1, y_2), y_3).$$

It is then clear that $\Omega_{12}h = (\Omega\pi_{12}h) \times \pi_3 h$. Hence $[(\Omega\pi_{12}h) \times \pi_3 h](X) \subset T^{2n} \vee T^{2n}$ so that $(\Omega\pi_{12}h) \times \pi_3 h$ is already normalized.
Then

$$(\alpha < \pi_{12}h > \beta) < \Omega_{12}h > \gamma = \Omega\Omega_{12}h.$$

Similarly

$$\alpha < \Omega_{23}h > (\beta < \pi_{23}h > \gamma) = \Omega\Omega_{23}h.$$

Since $\Omega\Omega_{12} = \Omega\Omega_{23}$, it follows that

$$(\{\alpha\} + \{\beta\}) + \{\gamma\} = \{\alpha\} + (\{\beta\} + \{\gamma\}).$$

(c) Existence of identity. Let e denote the map of (X, A) into (T^{2n}, q) defined by $e(x) = q$ for all $x \in X$. If $\alpha : (X, A) \to (T^{2n}, q)$ is arbitrary, $(e \times \alpha)(X) \subset T^{2n} \vee T^{2n}$ so $e \times \alpha$ is normalized. Hence, $\{e\} + \{\alpha\} = \{e < e \times \alpha > \alpha\} = \{\alpha\}$, so $\{e\}$ is an identity.
(d) Existence of inverses. Let $\alpha : (X, A) \to (T^{2n}, q)$. By [$D$] Lemma 2.6, there exists a normalization α' of α such that rel $\alpha^{-1}(S^n \vee S^n)$. The reflection of S^n in the equatorial plane of S^{n-1} is denoted by ρ_n, and $\rho_n \otimes \rho_n$ denotes the sum of the reflection of $S^n \times p$ and $p \times S^n$. It will be shown that $\{\alpha'\} + \{\rho_n \otimes \rho_n(\alpha')\} = \{e\}$. Let $\theta^+ : (S^n \times p \times I \cup p \times S^n \times I, p \times p \times I) \to (S^n \times p \cup p \times S^n, p \times p)$ be a contraction of $E^n_+ \times p \cup p \times E^n_+$ over itself into $p \times p$. Then θ^+_1 maps $(S^n \times p \cup p \times S^n, E^n_+ \times p \cup p \times E^n_+)$ into $(S^n \times p \cup p \times S^n, p \times p)$. We see that $\theta^+_1 \alpha' \simeq \theta^+_0 \alpha' = \alpha'$ and $\theta^+_1(\rho_n \otimes \rho_n(\alpha')) \simeq \rho_n \otimes \rho_n(\alpha')$. Therefore, $\{\alpha'\} + \{\rho_n \otimes \rho_n(\alpha')\} = \{\theta^+_1 \alpha'\} + \{\theta^+_1 \rho_n \otimes \rho_n(\alpha')\}$. Let $M_1 = \alpha'^{-1}(E^n_+ \times p \cup p \times E^n_+)$ and $M_2 = \alpha'^{-1}(E^n_- \times p \cup p \times E^n_-)$, Then $\theta^+_1 \alpha'$ maps M_1 into $p \times p$ and $\theta^+_1(\rho_n \otimes \rho_n(\alpha'))$ maps M_2 into $p \times p$. Hence, $\theta^+_1 \alpha' \times \theta^+_1(\rho_n \otimes \rho_n(\alpha'))$ maps X into $T^{2n} \vee T^{2n}$ so is normalized.
Then

$$\{\theta^+_1 \alpha'\} + \{\theta^+_1(\rho_n \otimes \rho_n(\alpha'))\} = \{\Omega[\theta^+_1 \alpha' \times \theta^+_1(\rho_n \otimes \rho_n(\alpha'))]\} \quad \text{and}$$

$$\Omega[\theta^+_1 \alpha' \times \theta^+_1(\rho_n \otimes \rho_n(\alpha'))](x) = \begin{cases} \theta^+_1(\rho_n \otimes \rho_n(\alpha'))(x) & \text{for } x \in M_1 \\ \theta^+_1 \alpha'(x) & \text{for } x \in M_2. \end{cases}$$

It follows that $\Omega[\theta^+_1 \alpha' \times \theta^+_1(\rho_n \otimes \rho_n(\alpha'))] \mid M_1 = \theta^+_1(\rho_n \otimes \rho_n(\alpha')) \mid M_1 \simeq \rho_n \otimes \rho_n(\alpha') \mid M_1$ and $\Omega[\theta^+_1 \alpha' \times \theta^+_1(\rho_n \otimes \rho_n(\alpha'))] \mid M_2 = \theta^+_1 \alpha' \mid M_2 = \alpha' \mid M_2$, and the two homotopies agree on $M_1 \cap M_2$. Define

$$h:(X, A)\to(T^{2n}, q)$$

by

$$h(x)=\begin{cases}\rho_n\otimes\rho_n(\alpha')(x) & \text{for } x\in M_2\\ \alpha'(x) & \text{for } x\in M_1.\end{cases}$$

Then $\{\theta_1^+\alpha'\}+\{\theta_1^+(\rho_n\otimes\rho_n(\alpha'))\}=\{h\}$. Since $h(X)\subset E_-^n\times p\cup p\times E^n$, it follows that $h\simeq e$, so $\{\alpha'\}+\{\rho_n\otimes\rho_n(\alpha')\}=\{e\}$.
The group whose existence was proved in Theorem 2.2 is called the $2n$th torus cohomotopy group of (X, A).

References

[1] K. Aoki: Note on deformation retract, Proc. Japan Acad., **30**, 538-541 (1954).
[2] R. H. Fox: Torus homotopy groups, Proc. Nat. Acad. Sci., **31**, 71-74 (1944).
[3] R. H. Fox: Homotopy groups and torus homotopy groups, Ann. Math., **49** (1948).
[4] E. Spanier: Borsuk's cohomotopy groups, Ann. Math., **50**, 203-245 (1949)