# 179. A Note on the General Metrization Problem

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A remarkable and important class of topological spaces was discovered by J. Dieudonné (3). He has called it paracompact space. An important result was obtained by A. H. Stone (9), who proved that every metric space is paracompact. The result of A. H. Stone led Yu. Smirnov (8) to find an interesting topological condition of the general metrization problem.[1)]

Hereafter all spaces will be understood to be Hausdorff spaces. A family $\alpha$ of open sets is called $\sigma$-locally finite, if $\alpha=\bigcup_{i=1}^{\infty}\alpha_i$ such that each $\alpha_i$ is locally finite. Then the result obtained by Yu. Smirnov (8) is the following

*Theorem. A necessary and sufficient condition that a regular topological space be metrizable is that it has an open $\sigma$-locally finite basis for the open sets.*

In this Note, we shall give a simple proof of Theorem.

From the results of A. H. Stone, the necessity of Theorem is clear. We turn now to prove the sufficiency of the condition. We first summarize some results needed in the proof.

Let $S$ be a regular topological space having a $\sigma$-locally finite basis for the open sets, then each subset of $S$ is regular, and, since a theorem of E. A. Michael (7), $S$ is paracompact and normal. Let $G$ be an open set in $S$, and let $x$ be a point of $G$, then there is an open set $O_n(x)\in\alpha_n$ for some $n$ such that $x\in O_n(x)\subset\overline{O_n(x)}\subset G$. Let $F_n$ be the closure of the union of $O_n$ contained in $\alpha_n$. Since the locally finiteness of $\alpha_n$, $F_n$ is the union of every closure of $O_n$. Therefore, we have $G=\bigcup_{h=1}^{\infty}F_n$. This shows that $S$ is perfectly normal. (Hence, since theorems of E. A. Michael (7) and J. Dieudonné (3), every subset of $S$ is paracompact and normal.)

The open covering of $S$ consisting of $\alpha_n$ plus $S$ has a locally finite refinement of open sets $\beta_n=\{U_{\alpha,n}\}$. Since $S$ is perfectly normal, we can find the continuous decomposition $\{\varphi_{\alpha,n}(x)\}$ of unity satisfying the following conditions

$$(1)\qquad \sum_{\alpha}\varphi_{\alpha,n}(x)=1 \quad\text{and}\quad 0\leq\varphi_{\alpha,n}(x)\leq 1,$$

1) The classical results and its proofs of the metrization problem was found in E. W. Chittenden (2), P. Lorenzen (6) and C. Kuratowski (5). On the other hand, R. H. Bing (1) has given a general condition by his new notions: If a regular topological space is perfectly screenable, then it is metrizable.

(2) $\varphi_{\alpha,n}(x)=0$ for $x \bar{\in} U_{\alpha,n}$ and $\varphi_{\alpha,n}(x) \neq 0$ for $x \in U_{\alpha,n}$.

To prove that $S$ is imbedded into a Hilbert space, an argument of C. H. Dowker (4) is available for us. Let $\psi_{\alpha,n}(x)=(\varphi_{\alpha,n}(x))^{\frac{1}{2}}/n$, then

$$\sum_{\alpha,n}(\psi_{\alpha,n}(x))^2=\sum\frac{1}{n^2}\varphi_{\alpha,n}(x)=\sum\frac{1}{n^2}<\infty.$$

Therefore $\{\psi_{\alpha,n}(x)\}$ for each $x \in S$ is a point of a Hilbert space $l^{(2)}$ with a metric $\rho$. Let $f$: $x\to\{\psi_{\alpha,n}(x)\}$ be a mapping from $S$ into $l^{(2)}$.

For $x$, $y \in S$ $(x \neq y)$, there are some $n$ and $U_{\alpha,n} \in \beta_n$ such that $x \in U_{\alpha,n}$, $y \bar{\in} U_{\alpha,n}$. Then $\psi_{\alpha,n}(x)>0$ and $\psi_{\alpha,n}(y)=0$. Hence we have $f(x) \neq f(y)$. This shows that $f$ is one-to-one mapping.

To prove that the mapping $f$ is continuous, let $x$ be a point of $S$ and let $\varepsilon$ be a given positive number. Then we can find an index $n_0$ such that $\sum_{n_0<n}\frac{1}{n^2}<\varepsilon$. For $\beta_n$ $(n \leq n_0)$, since each $\beta_n$ is locally finite, we can take a neighborhood $U(x)$ of $x$ which meets only a finite number of sets of each $\beta_n$ $(n \leq n_0)$. Let $U_{\alpha,n}$ $(n \leq n_0)$ be $U(x) \cap U_{\alpha,n} \neq 0$, then the number of such $U_{\alpha,n}$ is finite and $\psi_{\beta,n}(x)=0$ for $\alpha \neq \beta$, $n \leq n_0$. Therefore, since the continuity of $\psi_{\alpha,n}(x)$, we can choose a neighborhood $V(x)$ of $x$ such that

$$\sum_{n\leq n_0}(\psi_{\alpha,n}(x)-\psi_{\alpha,n}(y))^2<\varepsilon \text{ for } y \in V(x), \text{ and } U(x)\supset V(x).$$

Hence, $V(x) \ni y$ implies

$$\sum_{n,\alpha}(\psi_{\alpha,n}(x)-\psi_{\alpha,n}(y))^2 < 2\varepsilon.$$

This shows that $f$ is continuous.

We shall prove that $f^{-1}$ is continuous.

Let $f(S) \ni f(x)=\{\psi_{\alpha,n}(x)\}$ and take a neighborhood $U(x)$ of $x$, then there is a $U_{\alpha,n}$ such that $x \in U_{\alpha,n} \subset U(x)$ and we have $\psi_{\alpha,n}(x) > 0$. Let $\delta=\psi_{\alpha,n}(x)$, $\rho(f(x), f(y)) < \delta$ implies $\psi_{\alpha,n}(y) > 0$, hence $y \in U_{\alpha,n}$. Therefore, if $\rho(f(x), f(y))<\delta$, we have $y \in U(x)$. This shows that $f^{-1}$ is continuous. This completes the proof of the theorem.

## References


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