176. On Abhomotopy Group in Relative Case

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Introduction

The (n, r)-th abhomotopy group $\kappa_r^n(Y, y_0)$ of a space Y as base point $y_0 \in Y$ was defined by S. T. Hu as a generalization of Abe groups (M. Abe [1]). He showed that its algebraic structure is completely determined in terms of homotopy groups of Y, and that

$$(*) \qquad \kappa_r^n(Y, y_0) \approx \pi_{r+1}(Y^{S^{n-r-1}}, k_0) \quad r \geq 0,$$

where $Y^{S^{n-r-1}}$ is a mapping space consisting of all maps $f: S^{n-r-1} \to Y$ and topologized by compact open topology due to R. H. Fox (R. H. Fox [2]), and k_0 is a constant map: $k_0: S^{n-r-1} \to y_0$ (S. T. Hu[3]). In this paper, I shall show that the notion of abhomotopy group is relativized by using the same relation as (*). In this paper, we always denote by Y a given topological space, by Y_0 a subspace of Y and y_0 a reference point of Y_0. Then the *(m, n)-th relative abhomotopy group* $\kappa_n^m(Y, Y_0, y_0)$ of (Y, Y_0, y_0) is defined by

$$(**) \qquad \kappa_n^m(Y, Y_0, y_0) = \pi_m(Y^{E^n}\{S^{n-1}, Y_0\}, k_0) \quad m, n \geq 1,$$

where $Y^{E^n}\{S^{n-1}, Y_0\}$ is a mapping space consisting of all maps $f: E^n, S^{n-1} \to Y, Y_0$ and topologized by compact open topology. I shall show that, in § 2, its algebraic structure is completely determined by $\pi_{m+n}(Y, Y_0, y_0)$ and $\pi_m(Y_0, y_0)$. In § 1, for a preliminary of § 2, I describe a definition of relative homotopy groups which is obtained by a slightly modification of that of absolute homotopy groups given in the book "S. T. Hu [4] § 21".

§ 1. **Preliminary.** 1.1. Let I^{n+1} be the $(n+1)$-cube, and $\dot{I}^{n+1}$ be the boundary of I^{n+1} as usual. We use the following notations:

$$I^n = \{x^{n+1} = (x_1, \ldots, x_{n+1}) \in I^{n+1} \mid x_{n+1} = 0\},$$
$$J^n = \dot{I}^{n+1} - I^n,$$
$$P_n^n = \{x^{n+1} = (x_1, \ldots, x_{n+1}) \in I^{n+1} \mid x_n = 0\},$$
$$x_0 = (0, \ldots, 0) \in \dot{I}^{n+1}.$$

Let $\mathfrak{F} = Y^{J^n}\{\dot{I}^n, Y_0; x_0, y_0\}$ be the totality of maps $f: J^n, \dot{I}^n, x_0 \to Y, Y_0, y_0$. The maps f of $\mathfrak{F}$ are divided into disjoint homotopy classes relative to $\{\dot{I}^n, Y_0; x_0, y_0\}$. Denote by Ω the totality of these classes and by $[f]$ the class containing $f \in \mathfrak{F}$. Let f be a representative of an arbitrary element α of $\pi_n(Y, Y_0, y_0)$. Define a map $\mu f: J^n \to Y$ by taking for each $x^{n+1} = (x_1, \ldots, x_{n+1}) \in J^n$

$$\mu f(x^{n+1})=\begin{cases} f(x_1,\dots,x_{n-1},x_{n+1}) & \text{on } P^n_n \\ y_0 & \text{on } \overline{J^n-P^n_n}. \end{cases}$$

The map μf belongs to $\mathfrak{F}$, and $[\mu f]\in\Omega$ depends only on the element α. Then the correspondence $\alpha\to[\mu f]$ defines a one-to-one transformation $\mu^*:\pi_n(Y, Y_0, y_0)\to\Omega$ of $\pi_n(Y, Y_0, y_0)$ onto Ω. The proofs of this fact and the following theorems are parallel to that given in "S. T. Hu [4] § 21", and are omitted.

Theorem 1.1. *For an arbitrary map $f\in\mathfrak{F}$, $[f]=0$ if and only if f has an extension $f^*:I^{n+1}\to Y$ such that $f^*(I^n)\subseteqq Y_0$.*

Theorem 1.2. *Let $f, g\in\mathfrak{F}$ be two maps such that $f(\overline{J^n-P^n_n})=y_0=g(P^n_n)$ and let $h\in\mathfrak{F}$ be the map defined by*

$$h(x)=\begin{cases} f(x) & x\in P^n_n \\ g(x) & x\in\overline{J^n-P^n_n}. \end{cases}$$

Then $[h]=[f]+[g]$.

1.2. It is well known that each element ξ of $\pi_1(Y_0, y_0)$ induces an automorphism of $\pi_n(Y, Y_0, y_0)$, where $n\geq 2$ is any integer. Denote this automorphism by

$$\xi^*:\alpha\to\alpha^\xi \qquad \alpha\in\pi_n(Y, Y_0, y_0).$$

Let ω and f be representatives of $\xi\in\pi_1(Y_0, y_0)$ and $\alpha\in\pi_n(Y, Y_0, y_0)$ respectively. Define a map $g:J^n\to Y$ by taking for each point $x=(x_1,\dots,x_n,x_{n+1})\in J^n$,

$$g(x)=\begin{cases} f(x_1,\dots,x_{n-1},x_{n+1}) & \text{when } x_n=1 \\ \omega(x_n) & \text{when } 0<x_n<1 \\ y_0 & \text{when } x_n=0. \end{cases}$$

The class $[g]\in\Omega$ depends only on α and ξ, and

$$\alpha^\xi=-\mu^{*^{-1}}[g].$$

If, for every point $y_0\in Y_0$, the automorphisms ξ^* defined above are always identical, the space Y is called *n-simple relative to Y_0*.

§ 2. **Relative Abhomotopy Groups.** 2.1. Let Y^{I^n} be a mapping space consisting of all maps $f:I^n\to Y$ and topologized by compact open topology, and let $\mathfrak{F}^n(y_0)$ be a subspace of the space Y^{I^n}, which consists of all maps $f:I^n, I^{n-1}, J^{n-1}\to Y, Y_0, y_0$, where $n\geq 1$ is any integer. Denote by $\mathfrak{F}^n(Y_0)$ the union of all $\mathfrak{F}^n(y)$ for $y\in Y_0$, i.e. $\mathfrak{F}^n(Y_0)=\bigcup_{y\in Y_0}\mathfrak{F}^n(y)$. Since $\mathfrak{F}^n(y_0)$ is a subspace of the space $\mathfrak{F}^n(Y_0)$, we have the following homotopy sequence,

$$(1)\qquad \longrightarrow\pi_m(\mathfrak{F}^n(y_0), k_0)\xrightarrow{i_m}\pi_m(\mathfrak{F}^n(Y_0), k_0)\xrightarrow{j_m}\pi_m(\mathfrak{F}^n(Y_0), \mathfrak{F}^n(y_0), k_0)$$
$$\xrightarrow{\partial_m}\pi_{m-1}(\mathfrak{F}^n(y_0), k_0)\longrightarrow\cdots$$

where k_0 is the constant map; $k_0: I^n \to y_0$. It is well known that

$$\pi_m(\mathfrak{F}^n(y_0), k_0) \approx \pi_{m+n}(Y, Y_0, y_0). \tag{2}$$

When $Y_0 = y_0$ and $\mathfrak{F}^n(Y) = \smile_{y \in Y} \mathfrak{F}^n(y)$, $\pi_m(\mathfrak{F}^n(Y), k_0) = \kappa_{m-1}^{m+n}(Y, y_0)$. When $m=1$, $\pi_1(\mathfrak{F}^n(Y_0), k_0)$ identical with the group $\sigma_{n+1}(Y, Y_0, y_0)$ which was defined by H. Uehara in his paper [5]. We denote by $\kappa_n^m(Y, Y_0, y_0)$ the homotopy group $\pi_m(\mathfrak{F}^n(Y_0), k_0)$ and call it the (m, n) *-th relative abhomotopy group of* (Y, Y_0, y_0). It is obvious that this definition is identical with (**) in the introduction. In the sequel, we shall study the algebraic structure of relative abhomotopy groups. First, we prove the following lemma.

Lemma 2.1. *The image of the boundary homomorphism* ∂_m *is only the neutral element, for every integer* $m \geq 2$.

(proof) A representative f of an arbitrary element α of $\pi_m(\mathfrak{F}^n(Y_0), \mathfrak{F}^n(y_0), k_0)$ is characterized by

$$f(x^m, x^n) = \begin{cases} y_0 & \text{on } J^{m-1} \times I^n \\ f(x_1, \ldots, x_{m-1}, 0, x^n) & \text{on } I^{m-1} \times I^n \\ \omega(x^m) \in Y_0 & \text{on } I^m \times J^{n-1} \\ \in Y_0 & \text{on } I^m \times I^{n-1}, \end{cases} \tag{3}$$

where $x^m = (x_1, \ldots, x_m) \in I^m$, $x^n \in I^n$. For this characterization, define the following two maps $g, h: \dot{I}^m \times I^n \smile I^m \times J^{n-1} = J^{m+n-1} \to Y$ by taking

$$g(x^m, x^n) = \begin{cases} f(x_1, \ldots, x_{m-1}, 0, x^n) & \text{on } I^{m-1} \times I^n \\ y_0 & \text{on } J^{m-1} \times I^n \smile I^m \times J^{n-1} \end{cases}$$

$$h(x^m, x^n) = \begin{cases} \omega(x^m) & \text{on } I^m \times J^{n-1} \\ y_0 & \text{on } I^{m-1} \times I^n \smile J^{m-1} \times I^n. \end{cases}$$

The maps g and h represent the element $\beta, \gamma \in \pi_{m+n}(Y, Y_0, y_0)$ respectively. From the definition, $\beta = \partial_m \alpha$. Let f_0 be the partial map: $f_0 = f \mid \dot{I}^m \times I^n \smile I^m \times J^{n-1}$. Then by Theorem 1.2, $[f_0] = [g] + [h]$. Since f_0 has an extension f to $I^m \times I^n$, and since h has an extension $h^*: I^m \times I^n \to Y$ such that

$$h^*(x^m, x^n) = h(x^m) \quad \text{on} \quad I^m \times I^n,$$

then $[f_0] = [h] = 0$ by Theorem 1.1. Hence $[g] = 0$, i.e. $\beta = \partial\alpha = 0$. This completes the proof.

By the lemma stated above and from the exactness of the homotopy sequence (1), the homomorphism i_m is isomorphic into and the homomorphism j_m is onto. Therefore, the group $\kappa_n^m(Y, Y_0, y_0)$ contains a normal subgroup $\bar{\pi}_{m+n}$ isomorphic to $\pi_{m+n}(Y, Y_0, y_0)$.

A representative f of an arbitrary element α of $\kappa_n^m(Y, Y_0, y_0)$ is characterized by

$$(4) \qquad f(x^m, x^n) = \begin{cases} y_0 & \text{on } \dot{I}^m \times I^n \\ \omega(x^m) \in Y_0 & \text{on } I^m \times J^{n-1} \\ \in Y_0 & \text{on } I^m \times I^{n-1}, \end{cases}$$

where $\omega(x^m)=f(x^m, 0,\dots, 0)$. It is clear that $\omega(x^m) \in Y^{I^m}\{\dot{I}^m, y_0\}$. The element β of $\pi_m(Y_0, y_0)$ represented by ω depends only on α. By making correspondence α to β, we obtain a homomorphism:

$$p^* : \kappa_n^m(Y, Y_0, y_0) \to \pi_m(Y_0, y_0).$$

Conversely, for a representative ω of an element $\beta \in \pi_m(Y_0, y_0)$, the map $f_\omega : I^m \times I^n \to Y$ defined by

$$f_\omega(x^m, x^n) = \omega(x^n) \qquad \text{on} \quad I^m \times I^n$$

is a representative of an element α_ω of $\kappa_n^m(Y, Y_0, y_0)$ and $p^*(\alpha_\omega)=\beta$. The totality of such elements constructs a subgroup $\bar{\pi}_m$ of $\kappa_n^m(Y, Y_0, y_0)$ isomorphic to $\pi_m(Y_0, y_0)$. Therefore p^* is onto.

Lemma 2.2. *Kernel p^* = image i_m^*, for every integer $m \geq 1$.*

(proof) It is clear that $p^* i_m^* = 0$, conversely, we suppose that $p^*\alpha = 0$ for an element $\alpha \in \kappa_n^m(Y, Y_0, y)$. A representative f of α is characterized by (4). From the assumption $p^*\alpha=0$, there exists a homotopy $\omega_t : I^m \to Y (0 \leq t \leq 1)$ such that $\omega_0 = \omega$, $\omega_1 = y_0$. Define a homotopy $h_t : J^{m+n-1} = \dot{I}^m \times \dot{I}^n \smile I^m \times J^{n-1} \to Y (0 \leq t \leq 1)$ by

$$h_t(x^m, x^n) = \begin{cases} y_0 & \text{on } \dot{I}^m \times I^n \\ \omega_t(x^m) & \text{on } I^m \times J^{n-1}. \end{cases}$$

The homotopy h_t has an extension $h_t^* : I^m \times I^n \to Y$ such that $h_0^* = f$, $h_1^*(J^{m+n-1}) = y_0$, and $h_t^*(I^{m+n-1}) \subseteqq Y_0$. Obviously, the map h_1^* is a representative of an element $\gamma \in \pi_m(\mathfrak{F}^n(y_0), k_0)$. By the homotopy h_t^*, $i_m^*\gamma = \alpha$. This completes the proof.

By Lemmas 2.1 and 2.2, and from the exactness of the homotopy sequence (1), we have an isomorphism:

$$(5) \qquad \pi_m(\mathfrak{F}^n(Y_0), \mathfrak{F}^n(y_0), k_0) \approx \kappa_n^m(Y, Y_0, y_0) / \bar{\pi}_{m+n} \approx \pi_m(Y_0, y_0).$$

Summalizing, from the commutativity of the group $\kappa_n^m(Y, Y_0, y_0)$ for $m \geq 2$, we have the following theorem.

Theorem 2.3. *The group $\kappa_n^m(Y, Y_0, y_0)$ $(m \geq 1, n \geq 1)$ contains a normal subgroup $\bar{\pi}_{m+n}$ isomorphic to $\pi_{m+n}(Y, Y_0, y_0)$ and a subgroup $\bar{\pi}_m$ isomorphic to $\pi_m(Y_0, y_0)$. When $m \geq 2$, $\kappa_n^m(Y, Y_0, y_0)$ decomposes into the direct sum of two subgroups $\bar{\pi}_{m+n}$ and $\bar{\pi}_m$:*

$$(6) \qquad \kappa_n^m(Y, Y_0, y_0) = \bar{\pi}_{m+n} + \bar{\pi}_m \approx \pi_{m+n}(Y, Y_0, y_0) + \pi_m(Y_0, y_0).$$

2.2. When $m=1$, the group $\kappa_n^1(Y, Y_0, y_0)$ is a generalization of Abe groups. The group $\kappa_n^1(Y, Y_0, y_0)$ contains a normal subgroup $\bar{\pi}_{n+1}$ isomorphic to $\pi_{n+1}(Y, Y_0, y_0)$ and a subgroup $\bar{\pi}_1$ isomorphic to

$\pi_1(Y_0, y_0)$. In the group $\kappa_n^1(Y, Y_0, y_0)$, the operation of $\pi_1(Y_0, y_0)$ on $\pi_{n+1}(Y, Y_0, y_0)$ induces an inner automorphism:

$$\alpha^{\xi} = \bar{\xi}\alpha\bar{\xi}^{-1}, \tag{7}$$

where $\bar{\xi}$ is the element of π_1 such that $p^*(\bar{\xi}) = \xi$.

We prove the relation (7). From the definition of α^{ξ}, two maps f, g representing α and α^{ξ} are free homotopic relative to Y_0 with respect to the path ω representing ξ. Then there exists a map $F: I^{n+1}\times I \to Y$ such that

$$F(x^{n+1}, 1) = f(x^{n+1}), \quad F(x^{n+1}, 0) = g(x^n)$$
$$F(J^n, t) = \omega(t), \qquad F(I^n, t) \subseteq Y_0.$$

Define a homotopy $h_s: I^{n+1} \to Y (0 \leq s \leq 1)$ by

$$h_s(x^{n+1}) = \begin{cases} F(0, x_2, \ldots, x_{n+1}, 3x_1) = \omega(3x_1) & 0 \leq x_1 \leq \frac{1}{3}s \\ F\left(\frac{3x_1 - s}{3-2s}, x_2, \ldots, x_{n+1}, s\right) & \frac{1}{3}s \leq x_1 \leq 1 - \frac{1}{3}s \\ F(1, x_2, \ldots, x_{n+1}, 3-3x_1) = \omega(3-3x_1) & 1 - \frac{1}{3}s \leq x \leq 1. \end{cases}$$

Then $h_0 = g$ and $[h_1] = \bar{\xi}\alpha\bar{\xi}^{-1}$. By the homotopy h_s, $\alpha^{\xi} = \bar{\xi}\alpha\bar{\xi}^{-1}$. This establishes the relation (7). Thus, we have the following result.

Theorem 2.4. *The group $\kappa_n^1(Y, Y_0, y_0)$ $(n \geq 1)$ contains a normal subgroup $\bar{\pi}_{n+1}$ isomorphic to $\pi_{n+1}(Y, Y_0, y_0)$ and a subgroup $\bar{\pi}_1$ isomorphic to $\pi_1(Y_0, y_0)$, and is a split extention of $\pi_{n+1}(Y, Y_0, y_0)$ by $\pi_1(Y_0, y_0)$. A necessary and sufficient condition for Y to be $(n+1)$-simple relative to Y_0 is that $\kappa_n^1(Y, Y_0, y_0)$ decomposes into the direct product:*

$$\kappa_n^1(Y, Y_0, y_0) = \bar{\pi}_{n+1} \times \bar{\pi}_1 \approx \pi_{n+1}(Y, Y_0, y_0) \times \pi_1(Y_0, y_0). \tag{8}$$

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