3. *Notes on the Riemann-Sum*

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§ 1. Let $\{t_i(w)\}$ $i=1, 2,\dots$ be a sequence of independent random variables in a probability space (Ω, B, P) and each $t_i(w)$ has the uniform distribution in $[0, 1]$, that is

(1.1) $$F(x)=P(t_i(w)<x)$$

which is 1, x, or 0 according as $x>1$, $0\leqq x\leqq 1$ or $x<0$. For each w, let $t_i^{(n)}(w)$ denote the i-th value of $\{t_j(w)\}$ $(1\leqq j\leqq n)$ arranged in the increasing order of magnitude and let

(1.2) $$t_0^{(n)}(w)\equiv 0, \qquad t_{n+1}^{(n)}(w)\equiv 1, \qquad (n=1, 2,\dots).$$

Further let $f(t)$ $(-\infty<t<+\infty)$ be a Borel-measurable function with period 1 and belong to $L_1(0, 1)$.

Professor Kiyoshi Ito has recently proposed the problem: Does

(1.3) $$S_n(w)=\sum_{i=1}^{n} f(t_i^{(n)}(w))(t_i^{(n)}(w)-t_{i-1}^{(n)}(w))$$

converge to $\int_0^1 f(t)dt$ in any sense?

In this note, we consider the following translated Riemann-sum

(1.4) $$S_n(w, s)=\sum_{i=1}^{n} f(t_i^{(n)}(w)+s)(t_i^{(n)}(w)-t_{i-1}^{(n)}(w))$$

and prove the following

Theorem 1. *Let $f(t)$ be $L_2(0, 1)$-integrable and for any $\varepsilon>0$,*

(1.5) $$\left(\int_0^1 |f(t+h)-f(t)|^2 dt\right)^{1/2}=O\left(1\Big/\left|\log\frac{1}{|h|}\right|^{1+\varepsilon}\right) \quad (|h|\to 0).$$

Then for any fixed s, we have

$$P\left(\lim_{n\to\infty} S_n(w, s)=\int_0^1 f(t)dt\right)=1.$$

Remark. The w-set on which $S_n(w, s)\to\int_0^1 f(t)dt$ depends on s.

Theorem 2. *Let $f(t)$ be $L_1(0, 1)$-integrable and for an $\varepsilon>0$,*

(1.6) $$\int_0^1 |f(t+h)-f(t)|\, dt=O\left(1\Big/\left|\log\frac{1}{|h|}\right|^{1+\varepsilon}\right) \quad (|h|\to 0).$$

Then for any fixed w, except a w-set of probability zero, there exists a set $M_w\subset[0, 1]$ with measure 1 such that

$$\lim_{n\to\infty} S_n(w, s)=\int_0^1 f(t)dt \qquad (s\in M_w).$$

§ 2. By (1.1) and the independency of $\{t_i(w)\}$, it may be seen that

(2.1) $$P(\bigcup_{m\neq n}(t_m=t_n))=0.$$

On the other hand, we have

$$P(t_n(w)=t_n^{(n)}(w))=P(\smile [t_{v_1}<t_{v_2}<\cdots<t_{v_{n-1}}<t_n]),$$

where $\smile$ denotes the summation over all permutations of $(v_1, v_2, \dots v_{n-1})$ and v_i denotes an integer between 1 and $(n-1)$ such that $v_i \neq v_j$ if $i \neq j$.

For different permutations of $(v_1, v_2, \dots v_{n-1})$ the corresponding sets $[t_{v_1}<t_{v_2}<\cdots<t_{v_{n-1}}<t_n]$ are disjoint with each other and by the definitions of $\{t_i(w)\}$ $(1\leqq i)$, we have for any permutation of $(v_1, v_2, \dots v_{n-1})$

$$P(t_{v_1}<t_{v_2}<\cdots<t_{v_{n-1}}<t_n)=\int_0^1 dx_n \int_0^{x_n} dx_{n-1} \int_0^{x_{n-1}} dx_{n-2} \cdots \int_0^{x_2} dx_1 = \frac{1}{n!}.$$

Therefore we obtain

$$(2.2) \qquad P(t_n(w)=t_n^{(n)}(w))=1/n.$$

Let us put, for $i\leqq n$,

$$(2.3) \qquad d_{i,n}(w)=t_{j+1}^{(n)}(w)-t_i(w), \quad \text{if } t_i(w)=t_j^{(n)}(w) \qquad (j=1, 2, \dots n)$$

and

$$(2.3') \qquad d'_{i,n}(w)=t_i(w)-t_{j-1}^{(n)}(w), \quad \text{if } t_i(w)=t_j^{(n)}(w) \qquad (j=1, 2, \dots n).$$

Then we can write

$$(2.4) \qquad S_n(w, s)=\sum_{i=1}^n f(t_i(w)+s)d'_{i,n}(w).$$

Lemma 1. *We have, for* $0\leqq h\leqq 1$,

$$P(d_{i,n}(w)<h)=P(d'_{i,n}(w)<h)=1-(1-h)^n.$$

Proof. By the definition of $d_{i,n}(w)$, we have

$$P(d_{i,n}(w)<h)=P([d_{i,n}(w)<h]\cap[t_i(w)\leqq 1-h])+P(t_i(w)>1-h)$$
$$=\int_0^{1-h} P(d_{i,n}(w)<h \mid t_i(w)=x)dF(x)+h,$$

where $P(E|F)$ denotes the conditional probability of E under the hypothesis F. From the independency of $\{t_i(w)\}$, it follows that

$$P(d_{i,n}(w)<h \mid t_i(w)=x)=P(\bigcup_{\substack{j=1\\ j\neq i}}^{n}(x\leqq t_j(w)<x+h) \mid t_i(w)=x)$$
$$=P(\bigcup_{\substack{j=1\\ j\neq i}}^{n}(x\leqq t_j(w)<x+h))$$
$$=1-\prod_{\substack{j=1\\ j\neq i}}^{n}(1-P(x\leqq t_j(w)<x+h))=1-(1-h)^{n-1}.$$

Hence, we have

$$P(d_{i,n}(w)<h)=\int_0^{1-h}\{1-(1-h)^{n-1}\}dF(x)+h=1-(1-h)^n.$$

By the same way, we can show the second relation.

From the above lemma, it may be seen that

$$(2.5) \qquad P(\max_{1\leqq i\leqq n} d_{i,n}(w)\geqq 4\log n/n)\leqq \sum_{i=1}^n P(d_{i,n}(w)\geqq 4\log n/n)$$
$$=n(1-4\log n/n)^n=O(1/n^3) \qquad (n\to+\infty)$$

and

$$(2.5') \qquad P(\operatorname*{Max}_{1\leqq i\leqq n} d'_{i,n}(w)\geqq 4\log n/n)=O(1/n^3) \qquad (n\to+\infty).$$

By an easy estimation, it may be seen that

$$(2.6) \qquad \int_\Omega d'_{i,n}(w)dP=O(1/n), \quad \int_\Omega (d'_{i,n}(w))^2dP=O(1/n^2) \quad (n\to+\infty).$$

Lemma 2. *For every positive numbers x and y such that $x+y<1$, we have*

$$P([t_n(w)<x]\cap[d_{n,n}(w)<y])=x\{1-(1-y)^{n-1}\}.$$

Proof. We have, by the same way as the proof of Lemma 1

$$P([t_n(w)<x]\cap[d_{n,n}(w)<y])=\int_0^x P([d_{n,n}(w)<y]\,|\,[t_n(w)=z])dF(z)$$

$$=\int_0^x P(\bigcup_{i=1}^{n-1}(z\leqq t_i(w)<z+y)\,|\,t_n(w)=z)dF(z)$$

$$=\int_0^x P(\bigcup_{i=1}^{n-1}(z\leqq t_i(w)<z+y))dF(z)$$

$$=\int_0^x \{1-(1-y)^{n-1}\}dF(z)=x\{1-(1-y)^{n-1}\}.$$

The following lemma is well known.

Lemma 3. *Let $\{r_{i,n}\}$ be sequences of real numbers such that*

$$0=r_{0,n}<r_{1,n}<r_{2,n}<\cdots<r_{n,n}<r_{n+1,n}=1 \quad (n=1,2,\ldots)$$

and

$$\lim_{n\to\infty}\operatorname*{Max}_{0\leqq i\leqq n}(r_{i+1,n}-r_{i,n})=0.$$

Then, if $g(t)$ is $L_1(0,1)$-integrable and periodic with period 1, we have

$$\lim_{n\to\infty}\int_0^1 \Big|\sum_{i=1}^n g(r_{i,n}+s)(r_{i+1,n}-r_{i,n})-\int_0^1 g(t)dt\Big|\,ds=0.$$

§ 3. **Proof of Theorem 1.** From (2.4), we obtain

$$(3.1) \qquad S_n(w,s)-S_{n-1}(w,s)=d'_{n,n}(w)\{f(t_n(w)+s)-g_n(w,s)\}$$

where

$$(3.2) \qquad g_n(w,s)=\begin{cases}0, & \text{if } t_n(w)=t_n^{(n)}(w)\\ f(t_n(w)+d_{n,n}(w)+s), & \text{if } t_n(w)\neq t_n^{(n)}(w).\end{cases}$$

Since $t_n(w)+d_{n,n}(w)\leqq 1$, we have

$$\int_\Omega |S_n(w,s)-S_{n-1}(w,s)|\,dP$$

$$=\int_{E_1} d'_{n,n}(w)\,|f(t_n(w)+s)|\,dP+\int_{E_2} d'_{n,n}(w)\,|f(t_n(w)+s)-f(t_n(w)$$

$$+d_{n,n}(w)+s)|\,dP+\int_{E_3} d'_{n,n}(w)\,|f(t_n(w)+s)-f(t_n(w)+d_{n,n}(w)+s)|\,dP$$

$$=I_1+I_2+I_3$$

where $E_1=[t_n(w)=t_n^{(n)}(w)]$,

$E_2=[t_n(w)\neq t_n^{(n)}(w))\cap(t_n(w)+d_{n,n}(w)=1)]$,

and $E_3=[t_n(w)\neq t_n^{(n)}(w))\cap(t_n(w)+d_{n,n}(w)<1)]$.

Let us put $E_4=[d'_{n,n}(w)\geqq 4\log n/n]$.

By (2.2) and (2.5′), we have

$$I_1 \leqq \Big(\int_{E_1 \cap E_4} (d'_{n,n})^2 dP \Big)^{1/2} \Big(\int_{E_1 \cap E_4} |f(t_n(w)+s)|^2 dP \Big)^{1/2}$$
$$+ \Big(\int_{E_1 \cap (\Omega - E_4)} (d'_{n,n})^2 dP \Big)^{1/2} \Big(\int_{E_2 \cap (\Omega - E_4)} |f(t_n(w)+s)|^2 dP \Big)^{1/2}$$
$$= O(P^{1/2}(E_4)) + O(\log n / n P^{1/2}(E_1)) = O(\log n / n^{3/2}) \quad (n \to +\infty).$$

By (1.1) and the definition of $d_{n,n}(w)$, we obtain

$$P(E_2) \leqq P(\bigcup_{j=1}^{n-1} (t_j(w)=1)) = 0.$$

Therefore $I_2=0$. By (2.6), we have

$$I_3 \leqq \Big(\int_{E_3} (d'_{n,n})^2 dP \Big)^{1/2} \Big(\int_{E_3} |f(t_n(w)+d_{n,n}(w)+s) - f(t_n(w)+s)|^2 dP \Big)^{1/2}$$
$$= O(1/n) \Big(\int_{E_3} |f(t_n(w)+d_{n,n}(w)+s) - f(t_n(w)+s)|^2 dP \Big)^{1/2} \equiv O(1/n) \cdot I'_3,$$

say. From Lemma 2 and the definition of E_3, it is easily seen that

$$I'_3 = \Big\{ \iint_D |f(x+y+s) - f(x+s)|^2 (n-1)(1-y)^{n-2} dx dy \Big\}^{1/2}$$

where D denotes the domain ($0 \leqq x < 1$, $x+y<1$ and $0 \leqq y$). Hence we have, by (1.5),

$$I'_3 = \Big\{ \int_0^1 (n-1)(1-y)^{n-2} dy \int_0^{1-y} |f(x+y+s) - f(x+s)|^2 dx \Big\}^{1/2}$$
$$= \Big\{ \int_0^{\log(n-1)/(n-1)} (n-1)(1-y)^{n-2} dy \int_0^{1-y} |f(x+y+s) - f(x+s)|^2 dx$$
$$+ \int_{\log(n-1)/n-1}^1 (n-1)(1-y)^{n-2} dy \int_0^{1-y} |f(x+y+s) - f(x+s)|^2 dx \Big\}^{1/2}$$
$$= O\Big\{ \frac{1 - [1 - \log(n-1)/(n-1)]^{n-1}}{(\log n)^{2+2\varepsilon}} + \Big(1 - \frac{\log(n-1)}{n-1}\Big)^{n-1} \Big\}^{1/2} = O(1/(\log n)^{1+\varepsilon}).$$

Therefore, $$\sum_n \int_\Omega |S_n(w,s) - S_{n-1}(w,s)| dP < +\infty.$$

This proves that, for any fixed s,

(3.3) $$P(S_n(w,s) \ \text{converges}) = 1.$$

Hence, for the Proof of Theorem 1, it is sufficient to show that

(3.4) $$M_n = \int_\Omega | S_n(w,s) - \int_0^1 f(t) dt | dP = o(1) \qquad (n \to +\infty).$$

On the other hand, we have

(3.5) $$S_n(w,s) - \int_0^1 f(t) dt = \sum_{i=1}^{n} \int_0^{d'_{i,n}(w)} \{ f(t_i(w)+s) - f(t_i(w)+s-u) \} du$$
$$+ \int_0^{1-t_n^{(n)}(w)} f(t_n^{(n)}(w)+s+u) du.$$

Let us put $$E = [\operatorname*{Max}_{1 \leqq i \leqq n} d'_{i,n}(w) \geqq 4 \log n / n].$$

Then by (2.5′), it may be seen that

$$(3.6)\quad \int_E |S_n(w,s)-\int_0^1 f(t)dt|\,dP$$
$$\leqq \sum_{i=1}^n \int_E |f(t_i(w)+s)|\,dP+\int_E dP\,|\int_0^1 f(t)dt|$$
$$\leqq \sum_{i=1}^n P^{1/2}(E)\Big(\int_0^1 |f(t_i(w)+s)|^2 dP\Big)^{1/2}+P(E)\,|\int_0^1 f(t)dt| = O(1/n^{1/2})$$

and by (3.5)

$$(3.7)\quad V_n=\int_{\Omega-E} |S_n(w,s)-\int_0^1 f(t)dt|^2 dP$$
$$\leqq \Big(\sum_{i=1}^n \int_{\Omega-E} |S_n(w,s)-\int_0^1 f(t)dt|\,dP\int_0^{d'_{i,n}(w)} |f(t_i(w)+s)-f(t_i(w)+s-u)|\,du$$
$$+\int_{\Omega-E} |S_n(w,s)-\int_0^1 f(t)dt|\,dP\int_0^{1-t_n^{(n)}(w)} |f(t_n^{(n)}(w)+u+s)|\,du\Big)$$
$$\equiv \sum_{i=1}^n V_{i,n}+V'_n,$$

say. By (1.5) we have

$$(3.8)\quad V_{i,n}\leqq \int_{\Omega-E} |S_n(w,s)-\int_0^1 f(t)dt|\,dP\int_0^{4\log n/n} |f(t_i(w)+s)-f(t_i(w)+s-u)|\,du$$
$$=\int_0^{4\log n/n} du\int_{\Omega-E} |S_n(w,s)-\int_0^1 f(t)dt|\,|f(t_i(w)+s)-f(t_i(w)+s-u)|\,dP$$
$$\leqq \int_0^{4\log n/n} du\Big(\int_{\Omega-E} |f(t_i(w)+s)-f(t_i(w)+s-u)|^2 dP\Big)^{1/2} V_n^{1/2}$$
$$=O(V_n^{1/2}/n(\log n)^{\varepsilon}).$$

By the definitions of $t_n^{(n)}(w)$ and $d_{i,n}(w)$, it is seen that

$$(3.9)\quad V'_n=O\Big(\int_{\Omega-E} |S_n(w,s)-\int_0^1 f(t)dt|\,|1-t_n^{(n)}(w)|^{1/2} dP\Big)$$
$$=O\Big[V_n\Big(\int_{\Omega-E} |1-t_n^{(n)}(w)|\,dP\Big)\Big]^{1/2}$$
$$=O\Big(V_n\Big(\int_{1-t_n^{(n)}\geqq 4\log n/n} (1-t_n^{(n)})dP+\int_{1-t_n^{(n)}<4\log n/n} (1-t_n^{(n)})dP\Big)\Big)^{1/2}=O\Big(\frac{V_n^{1/2}(\log n)^{1/2}}{n^{1/2}}\Big).$$

By (3.7), (3.8) and (3.9), we get

$$(3.10)\qquad V_n^{1/2}=O(1/(\log n)^{\varepsilon})\qquad (n\to+\infty).$$

By (3.6), (3.7) and (3.10), it follows that

$$\int_\Omega \Big|S_n(w,s)-\int_0^1 f(t)dt\Big|\,dP=O(n^{-1/2})+O(1/(\log n)^{\varepsilon})=o(1)\qquad (n\to+\infty).$$

§4. Proof of Theorem 2. From (3.1) and (3.2), we have

$$\int_\Omega dP\int_0^1 |S_n(w,s)-S_{n-1}(w,s)|\,ds$$
$$=\int_{t_n^{(n)}\neq t_n} d'_{n,n}dP\int_0^1 |f(t_n(w)+s)-f(t_n(w)+d_{n,n}(w)+s)|\,ds$$

$$+\int_{t_n^{(n)}=t_n^{(n)}} d'_{n,n}dP\int_0^1 |f(t_n(w)+s)|\,ds \equiv J_1+J_2 \equiv J_1'+J_1''+J_2.$$

Let us put

$F_1=[t_n(w)\neq t_n^{(n)}(w)]$, $F_2=[d_{n,n}(w)\geqq 4\log n/n]$, $F_3=[d'_{n,n}(w)\geqq 4\log n/n]$.

Then we have, by (1.6), (2.5) and (2.6),

$$J_1'=\int_{F_1\cap F_2} d'_{n,n}dP\int_0^1 |f(t_n(w)+s)-f(t_n(w)+d_{n,n}(w)+s)|\,ds$$
$$\leqq\Big(2\int_0^1 |f(t)|\,dt\Big)P(F_2)=O(1/n^3) \qquad (n\to+\infty)$$

and
$$J_1''=\int_{F_1\cap(\Omega-F_2)} d'_{n,n}dP\int_0^1 |f(t_n(w)+s)-f(t_n(w)+d_{n,n}(w)+s|\,ds$$
$$=\Big(\int_{F_1\cap(\Omega-F_2)} d'_{n,n}dP\Big)O(1/(\log n)^{1+\varepsilon})$$
$$=O\Big(\frac{1}{(\log n)^{1+\varepsilon}}\int_\Omega d'_{n,n}dP\Big)=O(1/n(\log n)^{1+\varepsilon}) \qquad (n\to+\infty).$$

Also by (2.2) and (2.5′),

$$J_2=\int_{\Omega-F_1} d'_{n,n}dP\int_0^1 |f(t_n(w)+s)-f(t_n(w)+d_{n,n}(w)+s)|\,ds$$
$$\leqq\Big(2\int_0^1 |f(t)|\,dt\Big)\Big(\int_{F_3\cap(\Omega-F_1)} d'_{n,n}dP+\int_{(\Omega-F_3)\cap(\Omega-F_1)} d'_{n,n}dP\Big)$$
$$=O\Big(\frac{1}{n^3}+\frac{4\log n}{n}P(\Omega-F_1)\Big)=O(\log n/n^2) \qquad (n\to+\infty).$$

Therefore, it follows that

$$\sum_n\int_\Omega dP\int_0^1 |S_n(w,s)-S_{n-1}(w,s)|\,ds<+\infty$$

and this results for any fixed w, except a w-set of probability zero,

$$\lim_{n\to\infty} S_n(w,s)=S(w,s) \tag{4.1}$$

exists for almost all s, but the s-set depends on w.

On the other hand, by (2.5) and (2.5′), we have

$$P([\lim_{n\to\infty}\operatorname{Max}_{1\leqq i\leqq n} d_{i,n}(w)=0]\cap[\lim_{n\to\infty}\operatorname{Max}_{1\leqq i\leqq n} d'_{i,n}(w)=0])=1$$

and hence, by Lemma 3, it is seen that

$$\lim_{n\to\infty}\int_0^1\Big|S_n(w,s)-\int_0^1 f(t)dt\Big|\,ds=0 \tag{4.2}$$

holds for any fixed w except a w-set of probability zero. Using the Fatou's Lemma, we obtain, from (4.1) and (4.2),

$$\int_0^1\Big|S(w,s)-\int_0^1 f(t)dt\Big|\,ds=0 \tag{4.3}$$

which proves the theorem.