15. Some Remarks on Abhomotopy Groups

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1. Introduction. Abhomotopy groups has been introduced by S. T. Hu as a generalization of Abe groups (S. T. Hu [5]). Our purpose of the present paper is to show that abhomotopy groups can be treated as a special case of homotopy groups of pseudo fibre spaces. In the preceding paper [7], I defined abhomotopy groups of relative case. In latter part of this paper, I shall show that this groups is treated by the same method as above.

2. Pseudo Fibre Spaces. By a pseudo fibre space (X, p, B), we understand a collection of two spaces X, B and a continuous mapping $p: X \to B$ of X onto B which satisfy the "Lifting homotopy theorem" (p. 63, P. J. Hilton [3]; p. 443, J. P. Serre [8]). In this paper, we shall use the "Proposition 1" in p. 443 of J. P. Serre [8], which is equivalent to the "Lifting homotopy theorem". We recall that the homotopy sequence of a pseudo fibre space (X, p, B):

$$(1) \quad \longrightarrow \pi_{n+1}(B, b_0) \xrightarrow{d_{n+1}} \pi_n(X_0, x_0) \xrightarrow{i_n} \pi_n(X, x_0) \xrightarrow{p_n} \pi_n(B, b_0) \longrightarrow \cdots,$$

$n \geq 1$, is exact, where b_0 is a point of B, and x_0 is a point of the fibre $X_0 = p^{-1}(b_0)$ over b_0. In the sequel, we shall use these notations in these senses.

3. T-Operators. In the remainder of this paper, we assume that the total space X of a pseudo fibre space (X, p, B) is arcwise connected. J. P. Serre has proved in his paper [8] that $\pi_1(B)$ operate on the homology groups of the fibre X_0. By the same method, $\pi_1(X)$ operate on the homotopy groups of X_0. First, we prove the following theorem.

Theorem 1. Let (X, p, B) be a pseudo fibre space, x be a point of X and X_x be the fibre over $p(x) \in B$. Then, the collection of the n-th homotopy groups $\{\pi_n(X_x, x) \mid x \in X\}$ form a local system of groups in the space X. (For the definition of a local system of groups, refer to §13; S. T. Hu [6].)

(Proof) Let $\sigma: I \to X$ be a path joining two points x_0 and x_1. Let $f: I^n \to X$ be a map of an element α of $\pi_n(X_{x_1}, x_1)$. Define a map $F: I^n \times 0 \cup \dot{I}^n \times I \to X$ by taking for each $x^n \in I^n$, $t \in I$

$$F(x^n, t) = \begin{cases} f(x^n) & \text{on } I^n \times 0 \\ w(1-t) & \text{on } \dot{I}^n \times I. \end{cases}$$

Then the map $G = pF: I^n \times 0 \cup \dot{I}^n \times I \to B$ has the extention $G': I^n \times I \to B$ defined by $G'(x^n, t) = p\omega(1-t)$. By the "Proposition 1" in p. 443

of J. P. Serre [8], F has an extention F': $I^n \times I \to X$ such that $pF'=G'$. The map $\mathring{f}'=F' \mid I^n \times 1: I^n \to X$ is a representative of an element α' of $\pi_n(X_{x_0}, x_0)$. By making correspondence α to $\alpha'=\sigma_n^{**}(\alpha)$, we have an isomorphism $\sigma_n^{**}: \pi_n(X_{x_1}, x_1) \approx \pi_n(X_{x_0}, x_0)$. The detailed proof is similar as that of Theorem 13.6; S. T. Hu [6], and is omitted.

By this theorem, the fundamental group $\pi_1(X)$ acts as a group of left operator on the group $\pi_n(X_0)$, $n \geq 1$. Denote by ξ_n^{**} this operator on the group $\pi_n(X_0)$ induced by $\xi \in \pi_1(X)$, and we shall call the operator ξ_n^{**} a *T-operator* induced by ξ. The group $\pi_1(X_0)$ operates on the group $\pi_n(X_0)$ in the sense of S. Eilenberg. We shall call this usual operator ξ_n^* on the group $\pi_n(X_0)$ induced by $\xi \in \pi_1(X_0)$ an *E-operator* induced by ξ. Then, for $\xi \in \pi_1(X_0)$ and $\alpha \in \pi_n(X_0)$,

$$(i_1\xi)_n^{**}\alpha = \xi_n^*\alpha.$$

4. Direct Sum Decomposition Theorem. In this section, we assume that the pseudo fibre space has a cross section $\psi: B \to X$. For any integer $n \geq 1$, the n-th homotopy group $\pi_n(X)$ contains two subgroups M_n and N_n such that i_n maps $\pi_n(X_0)$ isomorphically onto M_n, p_n maps N_n isomorphically onto $\pi_n(B)$ and each element of $\pi_n(X)$ is uniquely representable as the product of an element of M_n and an element of N_n. For $n \geq 2$, we have the direct sum decomposition:

$$\pi_n(X) = M_n + N_n \approx \pi_n(X_0) + \pi_n(B) \tag{2}$$

(for example, see Theorem 27.6; S. T. Hu [6]). We recall that the cross section ψ induces an isomorphism ψ_n: $\pi_n(B) \to N_n$ such that $p_n\psi_n$ is the identity. Now, if all T-operators ξ_n^{**} on $\pi_n(X_0)$ for $\xi \in N_1$ are trivial, the pseudo fibre space (X, p, B) is said to be *n-cross simple with respect to the cross section* ψ. Then, by recalling that $\xi_1^{**}\alpha = \xi(i_1\alpha)\xi^{-1}$ for $\alpha \in \pi_1(X_0)$ and $\xi \in N_1$, we have the following theorem.

Theorem 2. The group $\pi_1(X)$ decomposes into the direct product of two subgroups M_1 and N_1, if and only if (X, p, B) is 1-cross simple with respect to the cross section ψ.

Theorem 3. $\pi_n(X, \psi(B)) \approx \pi_n(X_0)$, $n \geq 2$.

The natural isomorphism k_n: $\pi_n(X, \psi(B)) \to \pi_n(X_0)$ is defined as follows: Let f: $(I^n, \dot{I}^n, J^{n-1}) \to (X, \psi(B), x_0)$ be a map of α of $\pi_n(X, \psi(B))$. Define a map F: $(I^n \times 0, \dot{I}^n \times I, J^{n-1} \times I) \to (X, \psi(B), x_0)$ by taking for each $x^n=(x_1, \dots, x_n) \in I^n$, $t \in I$

$$F(x^n, t) = \begin{cases} f(x^n) & \text{for } t=0 \\ \psi p f(x_1, \dots, x_{n-1}, t) & \text{for } 0 \leq t \leq 1,\ x^n \in \dot{I}^n. \end{cases}$$

The map $pF=G$ is extended to the map G': $I^n \times I \to B$ defined by

$$G'(x^n, t) = pf(x_1, \dots, x_{n-1}, (1-t)x_n + t),\ x^n \in I^n,\ t \in I.$$

By "Proposition 1" in p. 443 of J. P. Serre [8], the map F extended to the map F': $I^n \times I \to X$ such that $pF'=G'$. Then, the partial map

$F'|I^n \times 1$ represents an element $\beta \in \pi_n(X_0)$ and $\beta = \kappa_n \alpha$. From this isomorphism, it is easy to prove the following theorem.

Theorem 4. The total space X is n-simple relative to $\psi(B)$, $n \geq 2$, if and only if the pseudo fibre space (X, p, B) is n-cross simple with respect to the cross section ψ.

5. *Abhomotopy Groups.* Let Y be an arcwise connected space. Denote by Y^{S^l} the function space of compact open topology consisting of all maps $f: S^l \to Y$ of l-sphere S^l into Y, where $l \geq 1$ is any integer. Denote by $Y_0^{S^l}$ the arcwise connected component of Y^{S^l} containing the constant map $k_{y_0}: S^l \to y_0$, a reference point of Y. Define a continuous mapping $p: Y_0^{S^l} \to Y$ of $Y_0^{S^l}$ onto Y by

$$pf = y \text{ when } f(s^l) = y,\ f \in S^l,$$

where $s^l = (1, 0, \dots, 0) \in S^l$. Then, the triple $(Y_0^{S^l}, p, Y)$ is a pseudo fibre space. (Refer to the proof of the "Proposition", p. 479; J.P. Serre [8].) Furthermore, the pseudo fibre space $(Y_0^{S^l}, p, Y)$ has a cross section $\psi: Y \to Y_0^{S^l}$ defined by

$$\psi(y) = k_y: S^l \to y \in Y,$$

which is called by us the natural cross section. Then the arguments in the preceding section are applicable. The fibre over a point $y_0 \in Y$ is the arcwise connected component $Y_0^{S^l}\{s^l, y_0\}$ of $Y^{S^l}\{s^l, y_0\}$ containing k_{y_0}, where $Y^{S^l}\{s^l, y_0\}$ is a function space of compact open topology consisting of all maps $f: (S^l, s^l) \to (Y, y_0)$. It is well known that $\pi_{m+l}(Y) \approx \pi_m(Y_0^{S^l}\{s^l, y_0\})$. Denote by ϕ_{m+l} its natural isomorphism. $\pi_m(Y_0^{S^l})$ is the abhomotopy group $k_{m-1}^{m+l}(Y)$ introduced by S. T. Hu [5]. Thus, from the general arguments stated in the preceding section, we have the direct sum relation:

$$\pi_m(Y_0^{S^l}) \approx \pi_{m+l}(Y) + \pi_m(Y) \qquad m \geq 2$$

of the abhomotopy group $\kappa_{m-1}^{m+l}(Y)$. For the proof of "Abe's Theorem", we shall prove the following lemma.

Lemma 5. Let ξ and α be elements of $\pi_1(Y)$ and $\pi_{m+l}(Y)$ respectively. Then,

$$(\psi_1 \xi)_m^{**} \phi_{m+l}(\alpha) = \phi_{m+l}(\xi_{m+l}^* \alpha),$$

where ξ_{m+l}^* is the E-operator of $\pi_{m+l}(Y)$ induced by ξ, and $(\psi_1 \xi)^{**}$ is the T-operator of $\pi_m(Y_0^{S^l}\{s^l, y_0\})$ induced by $\psi_1 \xi$.

(Proof) Let $f: (S^m, s^m) \to (Y_0^{S^l}, k_{y_0})$ and $\omega: (I, \dot{I}) \to (Y, y_0)$ be representatives of $\phi_{m+l}\,\alpha$ and ξ respectively. Define a map F: $S^m \times 0 \cup s^m \times I \to Y_0^{S^l}$ by taking for each $x^m \in S^m$, $t \in I$

$$F(x^m, t) = \begin{cases} f(x^m) & \text{on } S^m \times 0 \\ \psi\omega(1-t) & \text{on } s^m \times I. \end{cases}$$

The map F has an extention $F': S^m \times I \to Y_0^{S^l}$. Then, the partial map $f' = F'|S^m \times 1$ represents the element $(\psi_1 \xi)_m^{**} \phi_{m+l}(\alpha)$. The partial map $g'=G'|S^m\times S^l\times 1$ of the map $G': S^m\times S^l\times I\to Y$ defined by $G'(x^m, x^l, t)=F'(x^m, t)(x^l)$ is a map of the element $\xi^*_{m+l}\alpha$. This completes the proof.

Theorem 6. (*Abe's Theorem; M. Abe* [*1*]) *The Abe group* $\pi_1(Y_0^{S^l})$, $l\geq 1$, *decomposes into the direct product*:

$$\pi_1(Y_0^{S^l})=i_1\phi_{1+l}\pi_{1+l}(Y)\times\psi_1\pi_1(Y),$$

if and only if Y *is* $(1+l)$*-simple.*

The following theorem follows from Theorem 4 and the above lemma.

Theorem 7. The pseudo fibre $(Y_0^{S^l}, p, Y)$ *is m-cross simple with respect to the natural cross section* ψ, *if and only if* Y *is* $(m+l)$*-simple. Then, the space* $Y_0^{S^l}$ *is m-simple relative to* $\psi(\gamma)$ *if and only if* Y *is* $(m+l)$*-simple.*

6. *Abhomotopy Groups in Relative Case.* Let Y_1 be an arcwise connected subspace of Y. Let $Y^{E^l}\{S^{l-1}, Y_1\}$, $l\geq 1$, be a function space of compact open topology consisting of all maps $f: (E^l, S^{l-1})\to(Y, Y_1)$ of l-element E^l into Y such that $f(S^{l-1})\subseteq Y_1$. Denote by $Y_0^{E^l}\{S^{l-1}, Y_1\}$ the arcwise connected component of $Y^{E^l}\{S^{l-1}, Y_1\}$ containing the constant map $k_{y_0}: E^l\to y_0\in Y_1$. Define a continuous map $p: Y_0^{E^l}\{S^{l-1}, Y_1\}\to Y_1$ by

$$pf=y \qquad \text{when } f(s^{l-1})=y,\ f\in Y^{E^l}\{S^{l-1}, Y_1\}.$$

Then, the triple $(Y_0^{E^l}\{S^{l-1}, Y_1\}, p, Y_1)$ is a pseudo fibre space and has the natural cross section $\psi: Y_1\to Y_0^{E^l}\{S^{l-1}, Y_1\}$ defined by

$$\psi(y)=k_y: E^l\to y\in Y_1.$$

The fibre over a point $y_0\in Y_1$ is the arcwise connected component $Y_0^{E^l}\{S^{l-1}, s^{l-1}; Y_1, y_0\}$ of $Y^{E^l}\{S^{l-1}, s^{l-1}; Y_1, y_0\}$ containing the map k_{y_0}, where $Y^{E^l}\{S^{l-1}, s^{l-1}; Y_1, y_0\}$ is a function space of compact open topology consisting of all maps $f: (E^l, S^{l-1}, s^{l-1})\to(Y, Y_1, y_0)$. It is well known that $\pi_{m+l}(Y, Y_1)\approx\pi_m(Y_0^{E^l}\{S^{l-1}, s^{l-1}; Y_1, y_0\})$. Denote by ϕ_{m+l} its isomorphism. Thus, by the same arguments as in the section 5, we have the direct sum relation:

$$\pi_m(Y_0^{E^l}\{S^{l-1}, Y_1\})\approx\pi_{m+l}(Y, Y_1)+\pi_m(Y_1),\ m\geq 2 \text{ (Y. Inoue [7])},$$

and for $\xi\in\pi_1(Y_1)$ and $\alpha\in\pi_{m+l}(Y, Y_1)$,

$$(\psi_1\xi)^{**}_m\phi_{m+l}(\alpha)=\phi_{m+l}(\xi^*_{m+l}\alpha),$$

where ξ^*_{m+l} is the usual operator of $\pi_{m+l}(Y, Y_1)$ induced by ξ. From this relation, we have the following "Abe's Theorem" in relative case.

Theorem 8. (H. Uehara [9]) *The group* $\pi_1(Y_0^{E^l}\{S^{l-1}, Y_1\})$, $l\geq 1$, *decomposes into the direct product*:

$$\pi_1(Y_0^{E^l}\{S^{l-1}, Y_1\})=i_1\phi_{1+l}\pi_{1+l}(Y, Y_1)\times\psi_1\pi_1(Y_1),$$

if and only if Y *is* $(l+1)$*-simple relative to* Y_1.

7. *An Application.* For an application of the above results, we shall give a counter example of the "Theorem (4.1)" of S. T. Hu [4]. Let Y be an arcwise connected space. The r-th torus function space $\mathfrak{T}^r(Y)$, $r\geq 0$, is a function space of compact open topology consisting of all maps $f: I^r \to Y$ characterized by

$$f(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_r)=f(x_1,\dots,x_{i-1},1,x_{i+1},\dots,x_r)$$
$$i=1,2,\dots,r,$$

where $x^r=(x_1,\dots,x_r)\in I^r$ (R. H. Fox [2]). Let $\mathfrak{T}^r_0(Y)$ is an arcwise connected component of $\mathfrak{T}^r(Y)$ containing a constant map k_{y_0}: $I^r\to y_0\in Y$. $\mathfrak{T}^0_0(Y)$ is homeomorphic to Y and $\mathfrak{T}^r_0(Y)$ is homeomorphic to $\mathfrak{T}^{r-1}_0(Y)^{S^1}_0$. Consider the function space $\mathfrak{T}^r_0(Y)^{S^l}_0$, $l\geq 1$. The fundamental group $\pi_1(\mathfrak{T}^r_0(Y)^{S^l}_0)$ is isomorphically embedded in $\pi_1(\mathfrak{T}^{l+r}_0(Y))$. Then, from the algebraic structure of $\pi_1(\mathfrak{T}^{l+r}_0(Y))$ (§8, R. H. Fox [2]) and the "Abe's Theorem", we have the following theorem.

Theorem 9. The space $\mathfrak{T}^r_0(Y)$, $r\geq 0$, *is* $(l+1)$-*simple*, $l\geq 1$, *if and only if all Whitehead products* $[\alpha_k,\alpha_{k'}]$ *of elements* $\alpha_k\in\pi_{k+1}(Y)$ *and* $\alpha_{k'}\in\pi_{k'+1+l}(Y)$, *such that* $0\leq k+k'\leq r$, $0\leq k, k'\leq r$, *vanish.*

Then, by Theorem 7, we have the following theorem.

Theorem 10. The function space $\mathfrak{T}^r_0(Y)^{S^m}_0$ *is* $(l+1-m)$-*simple*, $l>m\geq 1$, *relative to* $\psi\mathfrak{T}^r_0(Y)$ *if and only if all Whitehead products* $[\alpha_k,\alpha_{k'}]$ *of elements* $\alpha_k\in\pi_{1+k}(Y)$ *and* $\alpha_{k'}\in\pi_{k'+1+l}(Y)$, *such that* $0\leq k+k'\leq r$, $0\leq k, k'\leq r$, *vanish, where* ψ *is the natural cross section.*

Let n be an even integer and let α be a generator of $\pi_n(S^n)$. The Whitehead product $[\alpha,\alpha]$ is a non zero element of $\pi_{2n-1}(S^n)$, in fact, the Hopf invariant of maps of $[\alpha,\alpha]$ is ± 2. Thus by Theorem 9, the space $\mathfrak{T}^r_0(S^n)$, $r\geq 2n-3>n\geq 4$, is not $(l+1)$-simple for any integer $n>l\geq 1$. Furthermore, by Theorem 10, the function space $\mathfrak{T}^r_0(S^n)^{S^m}_0$, $n-1>m\geq 1$, is not $(l+1-m)$-simple relative to $\psi\mathfrak{T}^r_0(S^n)$, for any integer $l>m\geq 1$. But the space $(\mathfrak{T}^r_0(S^n)^{S^m}_0)^{E^1}_0\{S^0,s^0;\psi\mathfrak{T}^r_0(S^n),k_{y_0}\}$ is q-simple for each integer $q\geq 1$. This fact is proved as follows: Let (X,p,B) be a pseudo fibre space with a cross section ψ. Denote by Ω the space $X^{E^1}_0\{S^0,s^0;\psi(B),x_0\}$. By Theorem 3, elements $\alpha\in\pi_m(\Omega)$ and $\xi\in\pi_1(\Omega)$ are represented by maps $f: (I^{m+1},\dot{I}^{m+1})\to(X,x_0)$ and $\omega: (I^2,\dot{I}^2)\to(X,x_0)$ respectively. Furthermore, we can suppose that $f(x_1,\dots,x_m,x_{m+1})=x_0$ for $\frac{1}{2}\leq x_{m+1}\leq 1$ and $\omega(x_1,x_2)=x_0$ for $0\leq x_2\leq\frac{1}{2}$. The map F: $(I^m\times I)^{\cdot}\times I\to X$ defined by

$$F(x_1,\dots,x_m,s,t)=\begin{cases} f(x_1,\dots,x_m,t) & \text{on } I^m\times\dot{I}\times I\\ \omega(s,t) & \text{on } \dot{I}^m\times I\times I\end{cases}$$

is extended to the map F': $I^m\times I\times I\to X$ defined by

$$F'(x_1,\dots,x_m,s,t)=\begin{cases} f(x_1,\dots,x_m,t) & \text{for } 0\leq t\leq\frac{1}{2}\\ \omega(s,t) & \text{for } \frac{1}{2}\leq t\leq 1.\end{cases}$$

Then, Ω is m-simple.

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