36. Vector-space Valued Functions on Semi-groups. II

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In an earlier Note (5),*) the author developed theory of vector-valued functions, especially almost periodic functions and ergodic functions on a semi-group G into a locally convex vector space E and proved the existence theorem of the mean value of ergodic function for some spaces.

In this Note, we shall consider a locally convex vector space E such that every ergodic function $f(x)$ has the mean $M(f)$. Therefore, there is an $M(f)$ of E such that, for any n.b.d. U,

$$M(f)-\frac{1}{n}\sum_{i=1}^{n} f(a_i d) \in U$$

and

$$M(f)-\frac{1}{m}\sum_{j=1}^{m} f(cb_j) \in U$$

for some $a_i (i=1,2,\dots,n)$, $b_j (j=1,2,\dots,m)$ and all c, d of G.

III. Invariant linear space of ergodic functions

The following propositions are clear.

Proposition 3.1. A constant function $f(x)\equiv f$ *has the mean* f: $M(f)=f$.

Proposition 3.2. If $f(x)$ *is ergodic, then* $\alpha f(x)$ *is ergodic and* $M(\alpha f)=\alpha M(f)$.

Definition 3. Let $\mathfrak{M}$ *be a set of ergodic functions.* $\mathfrak{M}$ *is said to be a left invariant linear set, if it satisfies the following conditions:*

(3) *for any element* a *of* G *and* $f(x)\in\mathfrak{M}$, $f(ax)\in\mathfrak{M}$,

(4) *for any reals,* α, β, *and* $f(x)$, $g(x)\in\mathfrak{M}$, $\alpha f(x)+\beta g(x)\in\mathfrak{M}$.

Theorem 8. Let $\mathfrak{M}$ *be a left invariant linear set of ergodic function, then*

(5) $$M_x(f(ax))=M_x(f(x)),$$

(6) $$M(\alpha f+\beta g)=\alpha M(f)+\beta M(g).$$

Proof. Let $f\in\mathfrak{M}$ and U any n.b.d., then there are elements $a_1, a_2,\dots, a_n$ and d of G such that

$$M_x(f(ax))-\frac{1}{n}\sum f(aa_i d)\in U.$$

Thus $M_x(f(ax))$ is U-left mean of $f(x)$. This proves (5).

We shall prove $M(f+g)=M(f)+M(g)$. Since the Propositions 1, 2, we have the equality (6). For a given n.b.d. U, we can find

*) Additional references are given in (5).

$a_1, a_2, \dots, a_n$ of G such that

$$(7) \qquad M(f)-\frac{1}{n}\sum f(a_i d) \in U.$$

Since $\mathfrak{M}$ is an invariant linear set, $\frac{1}{n}\sum g(a_i x)$ is ergodic, therefore, for a given n.b.d. V, there are $b_1, b_2, \dots, b_m$ such that

$$M_x\left\{\frac{1}{n}\sum g(a_i x)\right\}-\frac{1}{m}\sum_{j=1}^{m}\frac{1}{n}\sum_{i=1}^{n} g(a_i b_j d) \in V.$$

This shows that

$$M_x\left\{\frac{1}{n}\sum_{i=1}^{n} g(a_i x)\right\}=M_x(g(x)).$$

Hence

$$(8) \qquad M_x(g(x))-\frac{1}{mn}\sum_{i,j} g(a_i b_j d) \in V.$$

From (7), we have

$$(9) \qquad M_x(f(x))-\frac{1}{mn}\sum_{i,j} g(a_i b_j d) \in U.$$

Thus, (8) and (9) imply

$$M_x(f(x)+g(x))-\frac{1}{mn}\sum (f(a_i b_j d)+g(a_i b_j d)) \in U+V.$$

Hence, we have

$$M(f+g)=M(f)+M(g).$$

Definition 4. Let $f(x)$ be a function of G to E. $f(x)$ is said to be strong ergodic, if for any n.b.d. U, there is an element f of E such that

$$f-\frac{1}{n}\sum_{i=1}^{n} f(c a_i d) \in U$$

for some $a_1, a_2, \dots, a_n$ and any c, d of G.

We call f the *strong mean* of $f(x)$, and denote it by $\overline{\overline{M}}(f)$.

If G has unit, every strong ergodic function is ergodic.

If $f(x)$ is strong ergodic, then $f(cxd)$ is strong ergodic. Moreover, if $f(x)$, $g(x)$ are strong ergodic, then $\alpha f(x)+\beta g(x)$ is strong ergodic. We shall show $f(x)+g(x)$ is strong ergodic. For a given n.b.d. U, we can find f, g, and $a_1, \dots, a_n, b_1, \dots, b_m$ such that

$$f-\frac{1}{n}\sum_{i=1}^{n} f(c a_i d) \in U$$

and

$$g-\frac{1}{m}\sum_{j=1}^{m} g(c b_j d) \in U.$$

Hence

$$f-\frac{1}{mn}\sum_{i,j} f(c a_i b_j d) \in U$$

and

$$g-\frac{1}{mn}\sum_{i,j} g(ca_ib_jd) \in U.$$

Therefore, we have

$$f+g-\frac{1}{mn}\sum_{c,j} (f(ca_ib_jd)+g(ca_ib_jd)) \in U+U.$$

This shows that $f(x)+g(x)$ is strong ergodic.

Theorem 9. Let $f(x)$, $g(x)$ be strong ergodic functions, then

(10) *$\alpha f(x)+\beta g(x)$ is strongly ergodic, and*

$$\overline{M}(\alpha f+\beta g)=\alpha\overline{M}(f)+\beta\overline{M}(g),$$

(11) *$f(cxd)$ is strong ergodic, and*

$$\overline{M}_x(f(cxd))=\overline{M}_x(f(x)).$$

(12) *Any constant functions $f(x)\equiv f$ is strong ergodic and*

$$\overline{M}(f(x))=f.$$

IV. A mean value theorem of almost periodic functions

Suppose that a given semi-group G has unit element. Let $f(x)$ be a vector-space valued function on G.

For every n.b.d. U, there are two elements a_0, b_0 and a family of sets, $A_1,\dots,A_n$ of G such that

(13) $$\bigcup_{i=1}^{n} A_i = G.$$

(14) $c, d \in G$ and $x, y \in A_i$ imply

$$f(ca_0xb_0d)-f(ca_0yb_0d) \in U.$$

(15) $cxd, cyd \in A_i$ implies $f(x)-f(y) \in U$.

We shall consider a minimal decomposition $\{A_i\}_{i=1,2,\dots,n}$ of G. Such a decomposition exists. Therefore, we shall apply the combinatorial method by W. Maak (2), and we can take elements h_i of $A_{i(i=1,2,\dots,n)}$ such that

$$h_i \in ca_0A_{j_i}b_0d \quad \text{for all} \quad c, d \text{ of } G,\dots,$$

where A_{j_i} $(i=1,2,\dots,n)$ is some permutation of A_i. Therefore, we can find elements h_i' such that

$$h_i=ca_0h'_{j_i}b_0d, \qquad h'_{j_i} \in A'_{j_i}$$

for every i. For $a_i \in A_i$, we have

$$\frac{1}{n}\sum_{i=1}^{n} f(a_0)-\frac{1}{n}\sum f(ca_0a_ib_0d)=\frac{1}{n}\sum_{i=1}^{n}(f(a_i)-f(ca_0a_{j_i}b_0d))$$

$$=\frac{1}{n}\sum_{i=1}^{n}(f(a_i)-f(h_i))+\frac{1}{n}\sum_{i=1}^{n}\{f(h_i)-f(ca_0a_{j_i}b_0d)\}$$

$$\in\frac{1}{n}\sum_{i=1}^{n}U+\frac{1}{n}\sum_{i=1}^{n}(f(h_i)-f(ca_0a_{j_i}b_0d))$$

$$\subset U+\frac{1}{n}\sum_{i=1}^{n}(f(h_i)-f(ca_0a_{j_i}b_0d)).$$

On the other hand, since $h_i=ca_0h'_{j_i}b_0d$, we have

$$\frac{1}{n}\sum_{i=1}^{n}(f(h_i)-f(ca_0a_{j_i}b_0d)) \in U.$$

Hence

$$\frac{1}{n}\sum_{i=1}^{n}f(a_i)-\frac{1}{n}\sum f(ca_0a_ib_0d) \in 2U. \tag{16}$$

Putting here $c=d=1$, we have

$$\frac{1}{n}\sum_{i=1}^{n}f(a_0b_ib_0)-\frac{1}{n}\sum_{i=1}^{n}f(a_i) \in 2U. \tag{17}$$

Let $a_0b_ib_0=c_i$, then since (16), (17), we have

$$\frac{1}{n}\sum_{i=1}^{n}f(c_i)-\frac{1}{n}\sum_{i=1}^{n}f(cc_id) \in 4U$$

for every c, d of G. Therefore, we have the following

Theorem 10. Any almost periodic function on G is ergodic.

From the theorem for existence of mean value of any ergodic function, we have

Theorem 11. Any almost periodic function on a semi-group into a Banach space has one and only one mean.

Reference

5) K. Iséki: Vector-space valued functions on semi-groups. I, Proc. Japan Acad., **31**, 16–19 (1955).