33. Remarks on the Jordan-Hölder-Schreier Theorem*)

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The Jordan-Hölder-Schreier theorem, or shortly the J-H-S theorem, in lattices has been considered as the formulation of the J-H-S theorem for algebraic systems. But, A. W. Goldie has proved in his paper [3] the usual theorem on lengths of chains in modular lattices, using the Jordan-Hölder theorem for algebraic systems. In this note, the relations between these theorems will be more cleared. First, we shall show the J-H-S theorem for algebraic systems (§ 1). Next, considering a lattice L as the algebraic system with the composition $\cup$ only, we shall introduce a theorem for normal chains of L as the special case of the above theorem. And this theorem will be shown to be the usual J-H-S theorem in the lattice L (§ 2).

§ 1. **Algebraic Systems.** In this note we put the following conditions on the algebraic system A to keep out the complication:

0. All compositions are binary and single-valued, moreover any two elements may be composable by any composition.

I. A has a null-element e.

We denote by $\theta(B)$, $\varphi(B)$,... the congruences on a subsystem B of A. Moreover we denote by Θ the set of all congruences on all subsystems of A, i.e. $\Theta=\{\theta(B):B\subset A\}$.

Two congruences $\theta(B)$ and $\varphi(C)$ are called *weakly permutable* if and only if

$$(S(\theta(B\frown C))\,|\,\varphi(B\frown C))=(S(\varphi(B\frown C))\,|\,\theta(B\frown C)).$$

Moreover a congruence $\omega(B\frown C)$ is called a *quasi-join* of $\theta(B)$ and $\varphi(C)$, if and only if

i) $\omega(B\frown C)\geqq\theta(B\frown C)\smile\varphi(B\frown C)$ and

ii) $S(\omega(B\frown C))=S(\theta(B\frown C)\smile\varphi(B\frown C))$.

A subset Φ of Θ is called a *normal family*, when Φ has the following conditions:

i) Any two congruences in Φ are weakly permutable.

ii) For any congruences $\theta(B)$, $\varphi(C)$ in Φ, there exists a quasi-join $\omega(B\frown C)\in\Theta$ such that $[\omega(B\frown C)\,|\,\theta(B)]$, $[\omega(B\frown C)\,|\,\varphi(C)]\in\Phi$. Such a quasi-join $\omega(B\frown C)$ is called a *normal quasi-join*.

A normal chain

$$M=A_0\supset S(\theta_0(A_0))=A_1\supset\cdots\supset S(\theta_{r-1}(A_{r-1}))=A_r=N$$

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*) In this note, we shall use the theorems, the terms and the notations in [1] and [2], without the explanations.

is called an (M, N)-*Φ-normal chain,* when any $\theta_i(A_i)$ is a congruence in a given normal family Φ.

Theorem 1 (Schreier theorem for algebraic systems). *Let*

$$M=A_0\supset S(\theta_0(A_0))=A_1\supset\cdots\supset S(\theta_{r-1}(A_{r-1}))=A_r=N \text{ and}$$

$$M=B_0\supset S(\varphi_0(B_0))=B_1\supset\cdots\supset S(\varphi_{s-1}(B_{s-1}))=B_s=N$$

be any two (M, N)-Φ-normal chains. Then these chains can be refined by interpolation of terms $A_{i,j}=(A_i\frown B_j\,|\,\theta_i(A_i))=S([\omega_{i,j-1}(A_i\frown B_{j-1})\,|\,\theta_i(A_i)](A_{i,j-1}))$ *and* $B_{i,j}=(A_i\frown B_j\,|\,\varphi_j(B_j))=S([\omega_{i-1,j}(A_{i-1}\frown B_j)\,|\,\varphi_j(B_j)](B_{i-1,j}))$ *such that*

$$A_{i,j}/[\omega_{i,j}(A_i\frown B_j)\,|\,\theta_i(A_i)]\cong B_{i,j}/[\omega_{i,j}(A_i\frown B_j)\,|\,\varphi_j(B_j)],$$

where $\omega_{i,j}(A_i\frown B_j)$ are normal quasi-joins of $\theta_i(A_i)$ and $\varphi_j(B_j)$ respectively.

Proof. This theorem may be obtained by the same way as the proof of Theorem 7 in [2].

§ 2. **Lattices and J-systems.** Hereafter we assume that a lattice L has the least element 0 to keep out the complication. A lattice L is called a *J-system,* when L is considered as a algebraic system with the composition $\cup$ only. And the element 0 is considered as the null-element of the J-system L. A *J-congruence* means a congruence on a J-system, and an *ideal* means an ideal of lattices in the usual sense.

By Definition 1 and Theorem 1 in [2], any ideal of the J-system L is a normal sub-J-system, and conversely. Moreover the lower-J-congruence θ with respect to a normal sub-J-system N is defined by $x\overset{\theta}{\sim}y \rightleftarrows \exists n\in N\colon n\smile x=n\smile y$. In particular when N is a principal ideal $n/0$, θ is defined by $x\overset{\theta}{\sim}y\rightleftarrows n\smile x=n\smile y$.

A lower-J-congruence θ which is defined on a sub-J-system $m/0$ and whose normal sub-J-system is $n/0$, is called an $(m/0, n/0)$-*congruence.* In particular when n is $m/0$-modular, θ is called a *modular $(m/0, n/0)$-congruence,* or simply a *modular congruence.*

Lemma 1. *Let θ be an $(m/0, n/0)$-congruence. Then the quotient m/n is a representative system of the residue classes of $m/0$ with respect to θ, and $(m/0)/\theta$ is join-isomorphic to the quotient m/n.*

Proof. Let $x\in m/0$. Then $n\smile x=n\smile(n\smile x)$, i.e. $x\overset{\theta}{\sim}n\smile x$. Hence any class contains an element of m/n. On the other hand, let $x, y\in m/n$ and $x\neq y$. Then $n\smile x=x\neq y=n\smile y$. Hence x and y are not contained in a same class. Therefore m/n is a representative system of the residue classes, and $(m/0)/\theta$ is join-isomorphic to the quotient m/n.

Lemma 2. *Let $\theta(m/0)$ be a modular $(m/0, n/0)$-congruence, and a contained in L. Then $\theta(m\frown a/0)$ is a modular $(m\frown a/0, n\frown a/0)$-congruence.*

Proof. $n\frown a/0$ is evidently the normal sub-J-system with respect to $\theta(m\frown a/0)$, and by Theorem 2 in [1], $n\frown a$ is $m\frown a/0$-modular. Hence $\theta(m\frown a/0)\geqq$ the modular $(m\frown a/0, n\frown a/0)$-congruence φ. On the other hand, let $x, y\in m\frown a/0$ and $x\overset{\theta}{\sim}y$. Then $n\smile x=n\smile y$. Hence $(m\frown a)\frown(n\smile x)=(m\frown a)\frown(n\smile y)$. By the $m/0$-modularity of n, we get $(m\frown a\frown n)\smile x=(m\frown a\frown n)\smile y$. Hence $(n\frown a)\smile x=(n\frown a)\smile y$, i.e. $x\overset{\varphi}{\sim}y$. Hence $\theta(m\frown a/0)\leqq\varphi$. Therefore $\theta(m\frown a/0)$ is the modular $(m\frown a/0, n\frown a/0)$-congruence φ.

Lemma 3. *Let θ be a modular $(m/0, a/0)$-congruence, and φ a modular $(m/0, b/0)$-congruence. Then $(S(\theta)\,|\,\varphi)=(S(\varphi)\,|\,\theta)=a\smile b/0$.*

Proof. Let $x\in(S(\theta)\,|\,\varphi)$. Then there exists $a'\in S(\theta)=a/0$ such that $b\smile x=b\smile a'$. Hence by $b\smile a'\leqq b\smile a$, we get $x\in a\smile b/0$. Conversely, let $y\in a\smile b/0$. Then by the $m/0$-modularity of a, we get $((b\smile y)\frown a)\smile b=(b\smile y)\frown(a\smile b)=b\smile y$. Hence $y\overset{\varphi}{\sim}(b\smile y)\frown a\in a/0=S(\theta)$, i.e. $y\in(S(\theta)\,|\,\varphi)$. Therefore we get $(S(\theta)\,|\,\varphi)=a\smile b/0$. Similarly we get $(S(\varphi)\,|\,\theta)=a\smile b/0$.

Lemma 4. *The set Ψ of all modular congruences forms a normal family. In other words, let $\theta(m/0)$ be a modular $(m/0, a/0)$-congruence, and $\varphi(m'/0)$ a modular $(m'/0, a'/0)$-congruence. If $\omega(m\frown m'/0)=\theta(m\frown m'/0)\smile\varphi(m\frown m'/0)$, then $[\omega(m\frown m'/0)\,|\,\theta(m/0)]$ is a modular $((m\frown m')\smile a/0, (m\frown a')\smile a/0)$-congruence, and $[\omega(m\frown m'/0)\,|\,\varphi(m'/0)]$ is a modular $((m\frown m')\smile a'/0, (m'\frown a)\smile a'/0)$-congruence.*

Proof. By Lemma 2, $\theta(m\frown m'/0)$ is a modular $(m\frown m'/0, m'\frown a/0)$-congruence. Similarly $\varphi(m\frown m'/0)$ is a modular $(m\frown m'/0, m\frown a'/0)$-congruence. By Lemma 3, $\omega(m\frown m'/0)$ is a quasi-join of $\theta(m/0)$ and $\varphi(m'/0)$. By Theorem 4 in [2] and Theorem 5 in [1], $\omega(m\frown m'/0)$ is a modular $(m\frown m'/0, (m'\frown a)\smile(m\frown a')/0)$-congruence.

Now we shall prove that $[\omega(m\frown m'/0)\,|\,\theta(m/0)]$ is a modular $((m\frown m')\smile a/0, (m\frown a')\smile a/0)$-congruence ψ. First, by the $m/0$-modularity of a, $[\omega(m\frown m'/0)\,|\,\theta(m/0)]$ is defined on $(m\frown m')\smile a/0$ and its normal sub-J-system is $(m\frown a')\smile a/0$. Moreover by the $m\frown m'/0$-modularity of $(m'\frown a)\smile(m\frown a')$, it is clear that $(m\frown a')\smile a$ is $(m\frown m')\smile a/0$-modular. Hence $[\omega(m\frown m'/0)\,|\,\theta(m/0)]\geqq$ the modular congruence ψ. On the other hand, let x and y be congruent by $[\omega(m\frown m'/0)\,|\,\theta(m/0)]$. Then by the $m/0$-modularity of a and Theorem 1 in [1], we get

$$(*)\qquad [(x\smile a)\frown(m\frown m')]\smile a=x\smile a.$$

Hence $x\overset{\theta}{\sim}(x\smile a)\frown(m\frown m')$. Similarly $y\overset{\theta}{\sim}(y\smile a)\frown(m\frown m')$. Hence $(x\smile a)\frown(m\frown m')$ and $(y\smile a)\frown(m\frown m')$ are congruent by $[\omega(m\frown m'/0)\,|\,\theta(m/0)]$ and contained in the domain of $\omega(m\frown m'/0)$. Therefore $(x\smile a)\frown(m\frown m')$ and $(y\smile a)\frown(m\frown m')$ are congruent by $\omega(m\frown m'/0)$, i.e.

$$[(m'\frown a)\smile(m\frown a')]\smile[(x\smile a)\frown(m\frown m')]$$
$$=[(m'\frown a)\smile(m\frown a')]\smile[(y\smile a)\frown(m\frown m')].$$

Join a to both sides of this identity, and using $(*)$, we obtain

$$(m'\frown a)\smile(m\frown a')\smile(x\smile a)=(m'\frown a)\smile(m\frown a')\smile(y\smile a).$$

Hence $[(m\frown a')\smile a]\smile x=[(m\frown a')\smile a]\smile y$, i.e. $x\overset{\psi}{\equiv}y$. Hence $[\omega(m\frown m'/0)|\theta(m/0)]\leqq\psi$. Therefore $[\omega(m\frown m'/0)\,|\,\theta(m/0)]$ is the modular $((m\frown m')\smile a/0, (m\frown a')\smile a/0)$-congruence ψ. Similarly $[\omega(m\frown m'/0)\,|\,\psi(m'/0)]$ is the modular $((m\frown m')\smile a'/0, (m'\frown a)\smile a'/0)$-congruence.

Combining Theorem 1 and Lemma 4, we can immediately obtain the following

Theorem 2 (Schreier theorem for J-systems). *Let*

$$m/0=a_0/0\supset S(\theta_0(a_0/0))=a_1/0\supset\cdots\supset S(\theta_{r-1}(a_{r-1}/0))=a_r/0=n/0,$$
$$m/0=b_0/0\supset S(\varphi_0(b_0/0))=b_1/0\supset\cdots\supset S(\varphi_{s-1}(b_{s-1}/0))=b_s/0=n/0$$

be any two $(m/0, n/0)$-Ψ-normal chains. Then these chains can be refined by interpolation of terms $a_{i,j}/0=a_{i+1}\smile(a_i\frown b_j)/0$ and $b_{i,j}/0=b_{j+1}\smile(a_i\frown b_j)/0$ such that $(a_{i,j}/0)/\theta_{i,j}$ and $(b_{i,j}/0)/\varphi_{i,j}$ are join-isomorphic, where $\theta_{i,j}$ is the modular $(a_{i,j}/0, a_{i,j+1}/0)$-congruence, and $\varphi_{i,j}$ is the modular $(b_{i,j}/0, b_{i+1,j}/0)$-congruence.

By Lemma 1, the join-isomorphism between the quotients $a_{i,j}/a_{i,j+1}$ and $b_{i,j}/b_{i+1,j}$ is obtained from $(a_{i,j}/0)/\theta_{i,j}\cong(b_{i,j}/0)/\varphi_{i,j}$. Hence the quotients $a_{i,j}/a_{i,j+1}$ and $b_{i,j}/b_{i+1,j}$ are also isomorphic as lattices. Therefore translating Theorem 2 into the language of lattices, we can immediately obtain the following usual theorem in lattices:

Theorem 3 (Schreier theorem in lattices). *Let*

$$m=a_0>a_1>\cdots>a_r=n \text{ and } m=b_0>b_1>\cdots>b_s=n$$

be any two m/n-modular chains on 0. Then these chains can be refined by interpolation of terms $a_{i,j}=a_{i+1}\smile(a_i\frown b_j)$ and $b_{i,j}=b_{j+1}\smile(a_i\frown b_j)$ such that corresponding quotients $a_{i,j}/a_{i,j+1}$ and $b_{i,j}/b_{i+1,j}$ are isomorphic.

References

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