58. On the π-Regularity of Certain Rings

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In his paper [2][1] G. Azumaya introduced the notions of right, left, and strong π-regularities of rings (and of elements in a ring), and investigated connections between such types of rings, some of which had previously been studied by Kaplansky and others.[2] Recently one of the present authors obtained several properties on such rings under the assumption that the given ring is of bounded index (see [5]).

In the present note we shall generalize some results obtained in the papers remarked above by showing that they are applicable to some wider class of rings which contains, for example, rings with polynomial identities in the sense of Levitzki [3].

For the sake of convenience we insert here some definitions which are fundamental in our considerations:

An element a of a ring R is said to be *π-regular* in R if there exist an element x in R and a positive integer n such that $a^n x a^n = a^n$, and if there exist an x and an n such that $a^{n+1}x = a^n (xa^{n+1} = a^n)$ then a is said to be *right* (*left*) *π-regular*. An element which is right as well as left π-regular is said to be *strongly π-regular*. We say that R is a π-regular ring if every element of R is π-regular. Right, left, and, strongly π-regular rings are defined similarly. That a ring is *of bounded index* means that the least upper bound of all indices of nilpotent elements in the ring (=index of the ring) is finite.

1. Nil-ideals of bounded index. We consider first the following ring-property:

(*) A ring is nil and of bounded index.

Theorem 1. *The ring-property* (*) *is an additive $F\mathfrak{S}$-property:*[3]
(*E1*) *Each right* (*left*) *ideal in a* (*)-*ring is a* (*)-*right* (*left*) *ideal.*
(*E2*) *If A is a* (*)-*right* (*left*) *ideal in a ring R, then rA* (*Ar*) *is a* (*)-*right* (*left*) *ideal, where $r \in R$.*
(*E3*) *If $R^2=0$, then R is a* (*)-*ring.*
(*E4*) *If A is an ideal*[4] *of R such that $A^2=0$, then R is a* (*)-*ring if and only if R/A is so.*

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1) Numbers in brackets refer to the references at the end of this paper.
2) See, for example, the bibliography cited in [2].
3) See [4, §2].
4) The term "ideal" will mean a two-sided ideal throughout this paper.

(E5) If A is a $(*)$-right (left) ideal, then each left (right) ideal generated by an element in A is a $(*)$-left (right) ideal.
(F) Each homomorphic image of a $(*)$-ring is also a $(*)$-ring.
(A) The sum of two $(*)$-right (left) ideals is also a $(*)$-right (left) ideal.

Proof. (E1), (E3), (E4), and (F) are almost trivial.
(E2) Let $a^n=0$ for all $a \in A$. Then $(ra)^{n+1}=r(ar)^n a=0$.
(E5) $(ia+ra)^2=(i^2a+iar+ira+rar)a$, where $r \in R$ and i is an integer. Hence $(ia+ra)^{2(n+1)}=0$, where n is the index of A.
(A) Let n_1 and n_2 be the indices of $(*)$-right ideals A_1 and A_2 respectively. We may assume, without loss in generality, that each term in the expansion of $(a_1+a_2)^p$, $a_i \in A_i\,(i=1,2)$, is of the form $a_2^\alpha(\Pi\, a_1^\rho a_2^\sigma)a_1^\beta$, where $0 \leq \alpha < n_2$, $0 \leq \beta < n_1$, $0 < \rho < n_1$, $0 < \sigma < n_2$. There are at most $(n_1-1)(n_2-1)$ elements of the form $a_1^\rho a_2^\sigma$. Now we set $p=(n_1+n_2-2)\{f(n_1,(n_1-1)(n_2-1))+1\}$.[5] Then $(a_1+a_2)^p=0$, which implies that A_1+A_2 is a $(*)$-right ideal.

It is well known that a $(*)$-ring is semi-nilpotent.[6] Hence we obtain the next

Corollary. *The $(*)$-socle*[3] *of R is a sub-ideal of the Levitzki's radical.*

We consider here the sequence of ideals $B_\lambda=B_\lambda(R)$ which is defined by means of transfinite induction as follows:
(1) $B_0=0$.
(2) If λ is not a limit-ordinal, then B_λ is (uniquely) defined so that i) $B_\lambda \supseteq B_{\lambda-1}$ and ii) $B_\lambda/B_{\lambda-1}$ is the $(*)$-socle of $R/B_{\lambda-1}$.
(3) If λ is a limit-ordinal, then $B_\lambda=\bigcup_{\nu<\lambda}B_\nu$.

The limit ideal of this sequence will be denoted by $B^*=B^*(R)$.

Theorem 2. *B^* is a semi-nilpotent ideal and R/B^* contains no $(*)$-right ideal.*

Proof. The second part is obvious. To prove the first part, we assume that T is a subring of B_λ generated by $t_1,\dots,t_m$. If λ is not a limit-ordinal, then the semi-nilpotency of $B_\lambda/B_{\lambda-1}$ implies that $T^q \subseteq B_{\lambda-1}$ for some q. Clearly T^q is also finitely generated. On the other hand, in case λ is a limit-ordinal, there exists a non-limit-ordinal $\nu<\lambda$ such that $t_1,\dots,t_m$ are contained in B_ν. Hence we can complete the proof by transfinite induction.

Next, we consider $(*)$ as an ideal-property and construct the

5) Theorem 5 in [5] states: Let R be a nil-ring of bounded index n generated by a finite number, say g, of elements. Then R is nilpotent and we can take a positive integer m, which depends only upon n and g, such that $R^m=0$. We shall denote this integer by $f(n,g)$.

6) See J. Levitzki: On a problem of A. Kurosch, Bull. Amer. Math. Soc., **52**, 1033–1035 (1946), although the result stated there has been reestablished and sharpened elsewhere (cf. 5)).

following (uniquely determined) sequence of ideals $U_\lambda = U_\lambda(R)$:
(1) $U_0 = 0$.
(2) If λ is not a limit-ordinal, U_λ is the union of all ideals A in R such that $A/U_{\lambda-1}$ is a nil-ideal of bounded index.
(3) If λ is a limit-ordinal, then $U_\lambda = \bigcup_{\nu<\lambda} U_\nu$.

The limit ideal of this sequence will be denoted by $U^* = U^*(R)$ throughout this paper. By Theorem 1.1 of [1], U^* may be characterized as the intersection of all the ideals B such that R/B contains no nil-ideal of bounded index. As in Theorem 2, we can see that U^* is a semi-nilpotent ideal. Furthermore, to be easily verified, $U^*(R_n) = (U^*(R))_n$, where R_n denotes the $n \times n$ total matrix ring over R.

2. Local boundedness of index

Definition. A ring is said to be *of bounded index locally* if each ideal generated by a single element is of bounded index.

Let A be an ideal of R. If an element of A is π-regular (right π-regular, left π-regular) in R, then it is π-regular (right π-regular, left π-regular) already in A. Noting this fact, we obtain the following ([2, Theorem 1 and Theorem 5])

Lemma 1. *Let R be of bounded index locally. Then*
a) *every right (π-) regular element of R is (left whence) strongly (π-) regular.*
b) *The following conditions are equivalent to each other:*
  i) *R is π-regular,*
  ii) *R is right π-regular,*
  iii) *R is left π-regular,*
  iv) *R is strongly π-regular.*

The following lemma is a consequence of Theorem 1, (A) and [5, Theorem 5].

Lemma 2. *Let R be a π-regular ring of bounded index locally. Then R_n is also π-regular and of bounded index locally.*

Proof. Let $a = [a_{ij}]$ be a matrix in R_n. Then $(a) \subseteq \sum_{i,j} (a_{ij})_n$.[7] The rest of the proof is clear.

Lemma 3. *Let R be of bounded index locally. Then there exists the unique maximal π-regular ideal $\Pi(R)$.*

Proof. The proof is trivial by virtue of [5, Lemma 4].

3. Main results

Lemma 4. *If an element r in R is right π-regular modulo U^*, then it is virtually right π-regular.*

Proof. From our assumption, there exists the least ordinal λ for which there exist a positive integer p and an element x in R such that

7) (a), $a \in R$, signifies the ideal in R generated by a.

$$(1)\qquad r^{p+1}x - r^p = u \in U_\lambda.$$

By the minimality of λ, λ is not a limit-ordinal. If $\lambda \neq 0$, we denote by q the index of the ideal (u) modulo $U_{\lambda-1}$. Then, by (1), we obtain

$$(2)\qquad r^{pq+1}y - r^p \in (u),$$

where $y = x^{pq-p+1}$. Considering the q-th power of (2), we get $r^{pq+1}z - r^{pq} \in U_{\lambda-1}$ for some $z \in R$. But this contradicts the minimality of λ. Hence $\lambda = 0$.

Now we obtain the next

Theorem 3. *Let R/U^* be of bounded index locally. Then,*

a) the following eight conditions are equivalent to each other:

i) R is π-regular,	*i') R/U^* is π-regular,*
ii) R is right π-regular,	*ii') R/U^* is right π-regular,*
iii) R is left π-regular,	*iii') R/U^* is left π-regular,*
iv) R is strongly π-regular,	*iv') R/U^* is strongly π-regular.*

b) If R satisfies one of the conditions in a), then R_n is strongly π-regular and $R_n/U^*(R_n)$ is of bounded index locally.

Proof. a) is nothing but the combination of Lemma 1 and Lemma 4. b) follows from Lemma 1, Lemma 2, Lemma 4, and the fact that $(U^*(R))_n = U^*(R_n)$.

Corollary. *Let R be a ring with polynomial identities in the sense of Levitzki [3]. Then a) and b) in Theorem 3 are valid in this case too.*

Proof. In fact, R is of bounded index modulo the union of all nilpotent ideals.[8]

Theorem 4. *Let R/U^* be of bounded index locally. Then there exists the unique maximal π-regular ideal $\Pi(R)$.*

Proof. To be easily seen, $\Pi(R)/U^* = \Pi(R/U^*)$.

Theorem 5. *Let R/U^* be of bounded index locally. Then,*

a) $R/\Pi(R)$ contains no strongly π-regular ideals.

b) If I is an ideal in R, then $I \frown \Pi(R)$ is the unique maximal strongly π-regular ideal of I.

c) $(\Pi(R))_n$ is the unique maximal strongly π-regular ideal in R_n.

Proof. The proof is similar to that of Theorem 6 in [5].

References

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8) See [3, Theorem 1].