98. On Factor Set of the Third Obstruction

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The object of the present note[1] is to give the third obstruction theorem for mappings of a geometric complex K into a topological space Y such that

$$\pi_i(Y)=0 \quad \text{for } 0\leq i<n,\ n<i<q, \text{ and } q<i<r<2q-1,$$

along the line of Eilenberg-MacLane.[2]

For such a space Y we described previously[3] the cohomology class $\boldsymbol{k}_{n.q}^{r+1}$ of $H^{r+1}(\pi_n, \pi_q, \boldsymbol{k}_n^{q+1}; \pi_r)$[4] as a topological invariant if we pay no heed to the identification of the complexes $K(\pi_n, n, \pi_q, q, k)$, where $\boldsymbol{k}_n^{q+1}=\boldsymbol{k}_n^{q+1}(Y)$ is the Eilenberg-MacLane invariant of the space Y.

In this paper we shall introduce new operators $\boldsymbol{y}_{\Upsilon}$ and $\boldsymbol{y}_{\tau}$. And by making use of $\boldsymbol{k}_{n.q}^{r+1}{}_{\Upsilon}$, $\boldsymbol{k}_{n.q}^{r+1}{}_{\tau}$, we shall describe a factor set of the third obstruction of a map.

1. As a preliminary to the definition of the basic operators, we shall consider first certain maps.

We wish to classify simplicial maps T of a C.S.S. complex K in $K(\Pi, n, \Pi', q, k)$. Such a map determines a cocycle $x_n=T^{\#}b_n \in Z^n(K; \Pi)$ and a cochain $x_q=T^{\#}b_q \in C^q(K; \Pi')$, where b_n is the basic cocycle in $Z^n(\Pi, n, \Pi', q, k; \Pi) \cong Z^n(\Pi, n; \Pi)$ and b_q is the basic cochain in $C^q(\Pi, n, \Pi', q, k; \Pi')$ defined by setting

$$b_n(\phi, \psi)=\phi(\varepsilon_n), \qquad b_q(\phi, \psi)=\psi(\varepsilon_q).$$

Lemma 1. Given the complex $K(\Pi, n, \Pi', q, k)$ and the C.S.S. complex K, the rule $T\rightarrow(x_n, x_q)$ establishes a one to one correspondence between simplicial maps and pairs (x_n, x_q) satisfying the conditions:

$$x_n \in Z^n(K; \Pi),\ x_q \in C^q(K; \Pi'),\ kT(x_n)+\delta x_q=0.$$

The map T corresponding in this fashion to the pair (x_n, x_q) will be denoted by $T(x_n, x_q)$. Then $T(x_n, x_q)$ is characterized as

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1) Full details will appear in the Journal of the Institute of Polytechnics, Osaka City University.

2) S. Eilenberg and S. MacLane: On the groups $H(\Pi, n)$, III, Ann. Math., **60**, 513–557 (1954). Present note makes full use of the results and terminology of this paper.

3) K. Mizuno: On the minimal complexes, Jour. Inst. Polytech., Osaka City Univ., **5**, 41–51 (1954).

4) For the sake of brevity, we write in the following $\pi_n=\pi_n(Y)$, $\pi_q=\pi_q(Y)$, and $\pi_r=\pi_r(Y)$.

$$T(x_n, x_q)=\gamma[i_n\times i_q][T(x_n)\times T(x_q)]e$$

where i_n and i_q are natural inclusions.

For our future convenience, we now derive an explicit formula for the automorphism $\eta(\phi, \psi)=(\phi', \psi')$ such that

$$\eta: K(\Pi, n, \Pi', q, k)\to K(\Pi, n, \Pi', q, k)$$

$$\phi\equiv\phi' \quad \text{for any } (\phi, \psi) \text{ of } K(\Pi, n, \Pi', q, k).$$

According to Lemma 1, such a map η is represented as $T(b_n, b'_q)$ where b_n is the basic cocycle defined above and $b'_q=\eta^{\#}b_q$ is a cochain of $C^q(\Pi, n, \Pi', q, k; \Pi')$. Generally b'_q is different from b_q and induces a cocycle $h_q=b'_q-b_q$ of $Z^q(\Pi, n, \Pi', q, k; \Pi')$.

Lemma 2. Given the complex $K(\Pi, n, \Pi', q, k)$, *the rule* $\eta\to h_q$ *establishes a one to one correspondence between the chain homotopic class of* η *and the cohomology class of* h_q.

The map η corresponding in this fashion to the cocycle h_q will be denoted by $\eta(h_q)$, then $\eta(h_q)$ is characterized as

$$\eta(h_q)=T(b_n, b_q)\circ i_qT(h_q)$$

where $\circ$ is the internal product in the complex $K(\Pi, n, \Pi', q, k)$ and $T(b_n, b_q)$ is obviously the identity map.

If we replace x_q in the formula $T(x_n, x_q)$ by another x'_q, we have a cocycle $d_q=x'_q-x_q\in Z^q(K; \Pi')$, and successively, the map $T(x_n, x'_q)$ is represented by

$$T(x_n, x'_q)=T(x_n, x_q)\circ i_qT(d_q).$$

Therefore if we identify the complex $K(\Pi, n, \Pi', q, k)$ with the image of the automorphism η, we can identify $T(x_n, x'_q)$ with $T(x_n, x_q)$. Then we shall define $\tau(x_n)$ as the family of $T(x_n, x_q)$ where x_n is a fixed cocycle of $Z^n(K; \Pi)$ satisfying $kT(x_n)\sim 0$.

Lemma 3. The cocycles $x^1_n, x^2_n\in Z^n(K; \Pi)$ *such that* $kT(x^1_n)\sim 0\sim kT(x^2_n)$ *are cohomologous if and only if the families* $\tau(x^1_n)$, $\tau(x^2_n)$ *are chain homotopic.*[5]

Given two C.S.S. pairs (K, L_1), (K, L_2) and two cocycles $x_n\in Z^n(K, L_1; \Pi)$, $x_q\in Z^q(K, L_2; \Pi')$, we shall define a chain transformation

$$\gamma_{n.q}(x_n, x_q) : (K, L)\to K(\Pi, n, \Pi', q, k)$$

for each simplex σ ($\dim\sigma\leq 2q$) as

$$\gamma_{n.q}(x_n, x_q)\sigma=\gamma g[i_n\otimes i_q][R(x_n)\otimes R(x_q)]fe\sigma$$

where L is the union of the subcomplexes L_1, L_2.

Replacement of x_n or x_q by a cohomologous cocycle replaces $R(x_n)$ or $R(x_q)$ by a chain homotopic map, therefore, the homotopy class of the map $\gamma_{n.q}$ depends only on the cohomology classes $\boldsymbol{x}_n$, $\boldsymbol{x}_q$ of x_n, x_q respectively.

2. Take abelian groups Π, Π', and G, positive integers n, q, and

5) Namely, $\tau(x^1_n)$ and $\tau(x^2_n)$ contain $T(x^1_n, x^1_q)$, $T(x^2_n, x^2_q) : K\to K(\Pi, n, \Pi', q, k)$ respectively and $T(x^1_n, x^1_q)\simeq T(x^2_n, x^2_q)$.

r $(1<n<q<r<2q)$, a cohomology class $\boldsymbol{k} \in H^{q+1}(\Pi, n; \Pi')$, and a cohomology class $\boldsymbol{y} \in H^r(\Pi, n, \Pi', q, k; G)$ where k is a fixed cocycle of $\boldsymbol{y}$.

The γ-operator $\boldsymbol{y}_\gamma$ is defined for cohomology classes $\boldsymbol{x}_n \in H^n(K, L_1; \Pi)$ and $\boldsymbol{x}_q \in H^q(K, L_2; \Pi')$ by the formula

$$\boldsymbol{y}_\gamma(\boldsymbol{x}_n, \boldsymbol{x}_q) = \gamma_{n.q}(\boldsymbol{x}_n, \boldsymbol{x}_q)^* \boldsymbol{y};$$

this is an element of $H^r(K, L; G)$.

Let $\boldsymbol{x}_n \in H^n(K; \Pi)$ be a cohomology class whose representative cocycle x_n satisfies $kT(x_n) \sim 0$. The τ-operator $\boldsymbol{y}_\tau$ is defined for such a class $\boldsymbol{x}_n$ by the formula

$$\boldsymbol{y}_\tau(\boldsymbol{x}_n) = \tau(\boldsymbol{x}_n)^* \boldsymbol{y};$$

this is a family of elements of $H^r(K; G)$.

Theorem 4. For $\boldsymbol{x}_n \in H^n(K; \Pi)$ such that $\boldsymbol{k} \vdash \boldsymbol{x}_n = 0$, we can determine an element $\boldsymbol{y}_\tau(\boldsymbol{x}_n)$ of the factor group $H^r(K; G)$ mod $\boldsymbol{y}_\gamma(\boldsymbol{x}_n, H^q(K; \Pi')) + i_q^* \boldsymbol{y} \vdash H^q(K; \Pi')$ whose generator is represented by the formula $\boldsymbol{y}_\gamma(\boldsymbol{x}_n, \boldsymbol{x}_q) + i_q^* \boldsymbol{y} \vdash \boldsymbol{x}_q$, where $\boldsymbol{x}_q$ is any cohomology class going round the group $H^q(K; \Pi')$.

3. It is well known that the cohomology classes $\boldsymbol{k}_n^{q+1}(Y) \in H^{q+1}(\pi_n, n; \pi_q)$ and $\boldsymbol{k}_q^{r+1}(Y) \in H^{r+1}(\pi_q, q; \pi_r)$ attach to the space Y as topological invariants. And, it is obvious from our definition that $i_q^* \boldsymbol{k}_{n.q}^{r+1} = \boldsymbol{k}_q^{r+1}$.

In the identification of the complexes $K(\pi_n, n, \pi_q, q, k)$, the only essential part is the identification of the complex with the image of the automorphism η about which we considered above. Namely, we can recognize the invariant as the family $\{\eta(\boldsymbol{h}_q)^* \boldsymbol{k}_{n.q}^{r+1}\}$ of the classes of $H^{r+1}(\pi_n, n, \pi_q, q, k; \pi_r)$ for the fixed complex $K(\pi_n, n, \pi_q, q, k)$, where $\boldsymbol{h}_q$ is the cohomology class going round the $H^q(\pi_n, n, \pi_q, q, k; \pi_q) \cong H^q(Y; \pi_q)$. In other words, the invariant is the image $\tau(\boldsymbol{b}_n)^* \boldsymbol{k}_{n.q}^{r+1}$ of $\boldsymbol{k}_{n.q}^{r+1}$, and is an element of the factor group $H^{r+1}(\pi_n, \pi_q, \boldsymbol{k}_n^{q+1}; \pi_r)$ mod $\boldsymbol{k}_{n.q\,\gamma}^{r+1}(\boldsymbol{b}_n, H^q(\pi_n, \pi_q, \boldsymbol{k}_n^{q+1}; \pi_q)) + \boldsymbol{k}_q^{r+1} \vdash H^q(\pi_n, \pi_q, \boldsymbol{k}_n^{q+1}; \pi_q)$. In the following we shall denote this element simply as $\{\boldsymbol{k}_{n.q}^{r+1}\}$.

4. Let $f: K^n \smile L \to Y$ be a map extendible to a map $K^{n+1} \smile L \to Y$ with $f(K^{n-1}) = y_0$ which is the base point of Y, then a characteristic cocycle $a^n(f) \in Z^n(K; \pi_n)$ is determined as usual. If the second obstruction $\boldsymbol{z}^{q+1}(f) = 0$, the map f admits an extension $f': K^r \smile L \to Y$ and has an obstruction $c^{r+1}(f') \in Z^{r+1}(K, L; \pi_r)$. The cohomology class $\boldsymbol{z}^{r+1}(f')$ of this cocycle depends on the choice of the extension $f' \mid K^q \smile L$ as follows.

It follows from $\boldsymbol{z}^{q+1}(f) = 0$ that there is a cochain $a^q(f')$ in $C^q(K; \pi_q)$ satisfying $kT(a^n(f)) + \delta a^q(f') = 0$.

Theorem 5. Let $f_1, f_2: K^q \smile L \to Y$ be two extensions of the map f above and which are extendible to $K^{q+1} \smile L \to Y$. Then

$$\boldsymbol{z}^{r+1}(f_1) - \boldsymbol{z}^{r+1}(f_2) = \boldsymbol{k}_{n.q\,\gamma}^{r+1}(\boldsymbol{a}^n(f), \boldsymbol{a}^q(f_1, f_2)) + \boldsymbol{k}_q^{r+1} \vdash \boldsymbol{a}^q(f_1, f_2),$$

where $\boldsymbol{a}^q(f_1, f_2) \in H^q(K, L; \pi_q)$ *is the cohomology class of the cocycle* $a^q(f_1) - a^q(f_2)$, *and* $\boldsymbol{a}^n(f) \in H^q(K; \pi_n)$ *is the cohomology class of the cocycle* $a^n(f)$.

Theorem 6. Let $f: K^n \to Y$ *be a map extendible to a map* $K^{q+1} \to Y$, *then the third obstruction of* f *is determined as follows*:

$$\{\boldsymbol{z}^{r+1}(f)\} = \boldsymbol{k}^{r+1}_{n.q} {}_{\tau}\, \boldsymbol{a}^n(f).$$

5. We shall display a few properties of γ-and τ-operators in some special cases in the following.

a) If $n+q>r$, then $\boldsymbol{y}_\gamma(\boldsymbol{x}_n, \boldsymbol{x}_q) = 0$.

b) If $n+q=r$, then $\boldsymbol{y}_\gamma(\boldsymbol{x}_n, \boldsymbol{x}_q) = \boldsymbol{x}_n \smile \boldsymbol{x}_q$ where the cup product is taken relative to the pairing determined by $\boldsymbol{y}$. Especially if $\boldsymbol{y}$ is a representative class of the invariant $\{\boldsymbol{k}^{r+1}_{n.q}\}$, the cup product are paired by the Whitehead product.

c) If $n>2$, $r=q+1$ then the invariant $\{\boldsymbol{k}^{r+1}_{n.q}\}$ is determined as a coset of $H^{r+1}(\pi_n, \pi_q, \boldsymbol{k}^{q+1}_n; \pi_r)$ mod $Sq^2 H^q(\pi_n, \pi_q, \boldsymbol{k}^{q+1}_n; \pi_q)$. And the third obstruction is also determined as a coset of $H^{r+1}(K; \pi_r)$ mod $Sq^2 H^q(K; \pi_q)$.[6]

d) If $n=2$, $q=3$, $r=4$ then the third obstruction of a map $f: K^2 \to Y$ is determined as a coset of $H^5(K; \pi_4)$ mod $\boldsymbol{a}^2(f) \smile H^3(K; \pi_3) + Sq^2 H^3(K; \pi_3)$.[6]

6) Refer. N. Shimada and H. Uehara: On a homotopy classification of mappings of an $(n+1)$ dimensional complex into an arcwise connected topological space which is aspherical in dimensions less than $n(n>2)$, Nagoya Math. Jour., **3**, 67–72 (1951).