# 142. *Uniform Convergence of Fourier Series.* V

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1. R. Salem [1] proved the following theorem concerning uniform convergence of Fourier series.

**Theorem 1.** (i) *If* $f(x)$ *is continuous and*

$$(1)\qquad \frac{1}{h}\int_0^h (f(x+t)-f(x-t))dt=o\left(1\Big/\log\frac{1}{h}\right)\quad \text{as } h\to 0$$

*uniformly for all* $x$, *then the Fourier series of* $f(x)$ *converges uniformly everywhere.*

(ii) *If* $f(x)$ *is continuous in* $[a,b]$ *and the condition (1) is satisfied uniformly for* $x$ *in* $[a,b]$, *then the Fourier series of* $f(x)$ *converges uniformly in* $[a+\eta, b-\eta]$.

(iii) *If (1) holds uniformly in* $(a,b)$, *then the Fourier series of* $f(x)$ *converges almost everywhere in* $(a,b)$.

On the other hand, S. Izumi and G. Sunouchi [2] proved the following theorem concerning uniform convergence of Fourier series at a point.

**Theorem 2.** If

$$(2)\qquad f(t)-f(t')=o\left(1\Big/\log\frac{1}{|t-t'|}\right)\quad \text{as } t, t'\to x,$$

then the Fourier series of $f(t)$ converges uniformly at $t=x$.

S. Izumi-G. Sunouchi [2] and the author [3] proved theorems concerning uniform convergence of Fourier series at a point, under the conditions weaker than (2), with additional condition on the order of the Fourier coefficients of $f(x)$.

The object of this paper is to prove Theorem 1 by the method of R. Salem used in [2] and [3]. Further we prove theorems in [2] and [3], replaced uniform convergence at a point by that in an interval and their continuity conditions by those of type (1). We prove also similar theorems concerning ordinary convergence.

Finally we prove an improvement of another theorem of R. Salem [1], which gives the majorant of the partial sum of Fourier series.

2. We shall prove first Theorem 1, (i). Let $s_n(x)$ be the $n$th partial sum of the Fourier series of $f(x)$. Then it is sufficient to prove that $s_n(x)-f(x)=o(1)$, unif. for odd $n$. We put

$$s_n(x)-f(x)=\frac{1}{\pi}\int_0^\pi \varphi_x(t)\frac{\sin nt}{t}dt+o(1)=\frac{1}{\pi}\left[\int_0^{\pi/n}+\int_{\pi/n}^\pi\right]+o(1)$$
$$=I+J+o(1).$$

Since $f(x)$ is continuous,

$$|I| \leqq \frac{n}{\pi}\int_0^{\pi/n} |\varphi_x(t)|\, dt \leqq Cn\,\omega\left(\frac{\pi}{n}\right)\int_0^{\pi/n} dt \leqq C\omega(1/n) = o(1), \text{ unif.}$$

Concerning $J$, we get

$$\begin{aligned}
J &= \frac{1}{\pi}\int_{\pi/n}^{\pi} \varphi_x(t)\frac{\sin nt}{t}\,dt = \frac{1}{\pi}\sum_{k=1}^{n-1}\int_{k\pi/n}^{(k+1)\pi/n} \varphi_x(t)\frac{\sin nt}{t}\,dt \\
&= \frac{1}{\pi}\sum_{k=1}^{n-1}\int_0^{\pi/n} (-1)^k\varphi_x(t+k\pi/n)\frac{\sin nt}{t+k\pi/n}\,dt \\
&= \frac{1}{\pi}\sum_{k=1}^{(n-1)/2}\Bigg[\int_0^{\pi/n}\frac{\varphi_x(t+2k\pi/n)-\varphi_x(t+(2k-1)\pi/n)}{t+2k\pi/n}\sin nt\,dt \\
&\qquad -\frac{\pi}{n}\int_0^{\pi/n}\frac{\varphi_x(t+(2k-1)\pi/n)}{(t+2k\pi/n)(t+(2k-1)\pi/n)}\sin nt\,dt\Bigg] = \frac{1}{\pi}[J_1 - J_2],
\end{aligned}$$

say, and further we put

$$\begin{aligned}
J_1 &= \sum_{k=1}^{(n-1)/2}\Bigg[\int_0^{\pi/n}\frac{f(x+t+2k\pi/n)-f(x+t+(2k-1)\pi/n)}{t+2k\pi/n}\sin nt\,dt \\
&\qquad -\int_0^{\pi/n}\frac{f(x-t-2k\pi/n)-f(x-t-(2k-1)\pi/n)}{t+2k/n}\sin nt\,dt\Bigg] \\
&= J_{11} - J_{12}.
\end{aligned}$$

Then

$$\begin{aligned}
J_{11} &= \sum_{k=1}^{(n-1)/2}\Bigg[\int_0^{\pi/n}\frac{f(x+t+2k\pi/n)-f(x+t+(2k-1)\pi/n)}{2k\pi/n}\sin nt\,dt \\
&\qquad -\int_0^{\pi/n}\frac{f(x+t+2k\pi/n)-f(x+t+(2k-1)\pi/n)}{(t+2k\pi/n)2k\pi/n}\,t\sin nt\,dt\Bigg] \\
&= J_{11}^1 - J_{11}^2.
\end{aligned}$$

Let us now estimate $J_{11}^1$. By $L_k$ we denote the integral in $J_{11}^1$. Then

$$\begin{aligned}
L_k &= \int_0^{\pi/2n}[f(x+2k\pi/n+t)-f(x+(2k-1)\pi/n+t)]\sin nt\,dt \\
&\quad + \int_0^{\pi/2n}[f(x+2k\pi/n+(\pi/n-t))-f(x+(2k-1)\pi/n+(\pi/n-t))]\sin nt\,dt \\
&= \int_0^{\pi/2n}[f(x+2k\pi/n+t)-f(x+2k\pi/n-t)]\sin nt\,dt \\
&\qquad + \int_0^{\pi/2n}[f(x+(2k-1)\pi/n+t)-f(x+(2k+1)\pi/n-t)]\sin nt\,dt.
\end{aligned}$$

We put $\xi = x+2k\pi/n$, then we may write

$$\begin{aligned}
L_k &= \left[2\int_0^{\pi/2n} - \int_0^{\pi/n}\right](f(\xi+t)-f(\xi-t))\sin nt\,dt \\
&= 2L_{k1} + L_{k2}.
\end{aligned}$$

By integration by parts

$$L_{k1}=\Big[\sin nt\int_0^t (f(\xi+u)-f(\xi-u))du\Big]_0^{\pi/2n}$$

$$-n\int_0^{\pi/2n}\cos nt\,dt\int_0^t (f(\xi+u)-f(\xi-u))du$$

$$=o(1/n\log n)+o\Big(n\int_0^{\pi/2n}\Big(t\Big/\log\frac{1}{t}\Big)dt\Big)=o(1/n\log n),$$

and $L_{k2}$ is of the same order as $L_{k1}$. Accordingly we get

$$J_{11}^1=o\Big(\sum_{k=1}^{(n-1)/2}\frac{n}{k}\,\frac{1}{n\log n}\Big)=o(1).$$

On the other hand,

$$|J_{11}^2|\leqq\sum_{k=1}^{(n-1)/2}\int_0^{\pi/n}\frac{t\,|f(x+t+2k\pi/n)-f(x+t+(2k-1)\pi/n)|}{(t+2k\pi/n)2k\pi/n}dt$$

$$\leqq C\sum_{k=1}^{(n-1)/2}\omega\Big(\frac{\pi}{n}\Big)\frac{n^2}{k^2}\int_0^{\pi/n}t\,dt\leqq C\sum_{k=1}^{(n-1)/2}\frac{\omega(1/n)}{k^2}\leqq C\omega(1/n)=o(1),\quad\text{unif.}$$

Thus we have $J_{11}=o(1)$ unif. and also $J_{12}=o(1)$, unif.

Finally we estimate $J_2$.

$$|J_2|\leqq\frac{\pi}{n}\sum_{k=1}^{(n-1)/2}\int_0^{\pi/n}\frac{|\varphi_x(t+(2k-1)\pi/n)|}{(t+2k\pi/n)(t+(2k-1)\pi/n)}dt$$

$$\leqq\frac{2\pi}{n}\sum_{k=1}^{(n-1)/2}\int_0^{\pi/n}\frac{\omega(t+(2k-1)\pi/n)}{(t+(2k-1)\pi/n)^2}dt$$

$$=\frac{2\pi}{n}\Big[\sum_{k=1}^{[\sqrt{n}]}+\sum_{k=[\sqrt{n}]+1}^{(n-1)/2}\Big]\int_0^{\pi/n}\frac{\omega(t+(2k-1)\pi/n)}{(t+(2k-1)\pi/n)^2}dt.$$

If we denote by $M$ the maximum of $|f(x)|$, then the right hand side is less than

$$C\Big[\omega\Big(\frac{2\pi}{\sqrt{n}}\Big)\sum_{k=1}^{[\sqrt{n}]}\frac{1}{k^2}+M\sum_{k=[\sqrt{n}]+1}^{(n-1)/2}\frac{1}{k^2}\Big]\leqq C\Big[\omega\Big(\frac{1}{\sqrt{n}}\Big)+\frac{M}{\sqrt{n}}\Big]$$

which tends to zero as $n\to\infty$. Thus the theorem is proved.

3. We shall next prove that the condition (1) in Theorem 1 may be weakened when some order condition of the Fourier coefficients of $f(x)$ is added (cf. [2], [3]).

**Theorem 3.** *Let $0<\alpha<1$. If $f(x)$ is continuous and*

$$(3)\qquad \frac{1}{h}\int_0^h (f(x+t)-f(x-t))dt=o\Big(1\Big/\Big(\log\frac{1}{h}\Big)^\alpha\Big)\quad \text{as } h\to 0$$

*uniformly for all $x$ and further the nth Fourier coefficients of $f(x)$ are of order $O(e^{(\log n)^\alpha}/n)$, then the Fourier series of $f(x)$ converges uniformly everywhere.*

In order to prove this theorem, we put

$$s_n(x)-f(x)=\frac{1}{\pi}\int_0^\pi \varphi_x(t)\frac{\sin nt}{t}dt+o(1)$$

$$=\frac{1}{\pi}\left[\int_0^{\pi/n}+\int_{\pi/n}^{\pi e^{\beta(\log n)^\alpha}/n}+\int_{\pi e^{\beta(\log n)^\alpha}/n}^{\pi}\right]+o(1)=\frac{1}{\pi}[I+J+K]+o(1),$$

where $\beta$ is the least number larger than 1 such that $e^{\beta(\log n)^\alpha}/n-1$ is odd.

Since $f(x)$ is continuous, we have $I=o(1)$. $J$ is different from the $J$ in the proof of Theorem 1 only in the upper bound of summation, that is $n-1$ in $J$ in Theorem 1 is replaced by $e^{\beta(\log n)^\alpha}-1$. Hence similar estimation holds. The terms corresponding to $J_{11}^2$ and $J_2$ are of $o(1)$. The term corresponding to $J_{11}^1$ is

$$o\left(\sum_{k=1}^{e^{\beta(\log n)^\alpha}-1}\frac{1}{k}\frac{1}{(\log n)^\alpha}\right)=o(1);$$

thus $J=o(1)$. Finally

$$K=2\sum_{\nu=1}^{\infty}a_\nu\cos\nu x\int_{e^{\beta(\log n)^\alpha}/n}^{\pi}\cos\nu t\frac{\sin nt}{t}\,dt.$$

We can prove that $K=o(1)$ by the last condition of the theorem (cf. [3]).

Further we can prove in a similar manner the following theorems.

**Theorem 4.** *Let $\alpha>1$. If $f(x)$ is continuous and*

$$\frac{1}{h}\int_0^h(f(x+t)-f(x-t))dt=o\left(1\Big/\left(\log\log\frac{1}{h}\right)^\alpha\right)\quad as\ h\to 0$$

*uniformly for all $x$ and further the nth Fourier coefficients of $f(x)$ are of order $O(e^{(\log\log n)^\alpha}/n)$, then the Fourier series of $f(x)$ converges uniformly everywhere (cf. [3]).*

*If $\alpha=1$, then the conclusion holds when $O(e^{(\log\log n)^\alpha}/n)$ in the last condition is replaced by $O((\log n)^\gamma/n)$, $(\gamma>0)$ (cf. [2]).*

**Theorem 5.** *If $f(x)$ is continuous and*

$$\frac{1}{h}\int_0^h(f(x+t)-f(x-t))dt=o\left(1\Big/\psi\left(\frac{1}{h}\right)\right)\quad as\ h\to 0$$

*and if $f(x)$ is of class $\phi(n)$, then the Fourier series of $f(x)$ converges uniformly everywhere, where $\phi(n)=O(n)$, $\psi(n)=\log(n\theta(n)/\phi(n))$, and $\theta(n)$ are monotone increasing to infinity as $n\to\infty$ (cf. [3]).*

From Theorem 5 we get the following corollaries.

**Corollary 1.** *Let $0<\alpha<1$. If*

$$\frac{1}{h}\int_0^h(f(x+t)-f(x-t))dt=o\left(1\Big/\left(\log\log\frac{1}{h}\right)^\alpha\right)\quad as\ h\to 0$$

*and if $f(x)$ is of class $\phi(n)=n/e^{(\log\log n)^\alpha}$, then the Fourier series converges uniformly everywhere.*

**Corollary 2.** *Let $\alpha>0$ and $k$ be an integer $\geqq 3$. If*[1]

$$\frac{1}{h}\int_0^h(f(x+t)-f(x-t))dt=o\left(1\Big/\left(\log_k\frac{1}{h}\right)^\alpha\right)\quad as\ h\to 0$$

1) $\log(\log x)=\log_2 x$, $\log_k(\log x)=\log_{k+1}x$ $(k\geqq 2)$.

*and if $f(x)$ is of class $\phi(n)=n/e^{(\log_k n)^\alpha}$, then the Fourier series converges uniformly everywhere.*

**4.** We can prove Theorem 1, (ii), similarly to Theorem 1, (i). For,

$$s_n(x)-f(x)=\frac{1}{\pi}\int_0^{2\pi}\varphi_x(t)\frac{\sin nt}{t}dt+o(1)=\frac{1}{\pi}\left[\int_0^{\eta}+\int_{\eta}^{2\pi}\right]+o(1),$$

where the first integral is $o(1)$ uniformly as in the proof of Theorem 1, (i). The second integral is estimated as follows.

$$\left|\int_{\eta}^{2\pi}\frac{\varphi_x(t)}{t}\sin nt\,dt\right|$$

$$\leqq\frac{1}{2}\left[\int_{\eta}^{2\pi}\left|\frac{\varphi_x(t)}{t}-\frac{\varphi_x(t+\pi/n)}{t-\pi/n}\right|dt+\int_{\eta-\pi/n}^{\eta}\frac{|\varphi_x(t+\pi/n)|}{t+\pi/n}dt+\int_{2\pi-\pi/n}^{2\pi}\frac{|\varphi_x(t)|}{t}dt\right]$$

$$=\frac{1}{2}[K_1+K_2+K_3],$$

say. Then

$$K_1\leqq\int_{\eta}^{2\pi}\frac{|\varphi_x(t)-\varphi_x(t+\pi/n)|}{t}dt\leqq\frac{1}{\eta}\int_0^{2\pi}|\varphi_x(t)-\varphi_x(t+\pi/n)|\,dt$$

$$=\frac{1}{\eta}\left[\int_0^{2\pi}(|f(x+t)-f(x+t+\pi/n)|+|f(x-t)-f(x-t-\pi/n)|)dt\right].$$

By a familiar theorem the right side is of $o(1)$ uniformly as $n\to\infty$. Similarly $K_2+K_3$ is also of $o(1)$ uniformly.

We can generalize Theorems 3, 4, and 5 in the type of Theorem 1, (ii).

**5.** We shall prove Theorem 1, (iii), in the following form:

**Theorem 6.** *If (1) holds uniformly for all $x$, then the Fourier series of $f(x)$ converges at all Lebesgue points.*

In the proof of Theorem 1, (i), continuity condition of $f(x)$ is used in the estimation of $J_{11}^2$ and $J_2$ only. For the proof of Theorem 6, it is sufficient to prove that $J_{11}^2$ and $J_2$ are of $o(1)$ at Lebesgue points.

$$J_{11}^2=\sum_{k=1}^{(n-1)/2}\int_0^{\pi/n}\frac{f(x+t+2k\pi/n)-f(x+t+(2k-1)\pi/n)}{(t+2k\pi/n)2k\pi/n}t\sin nt\,dt,$$

$$|J_{11}^2|\leqq A\sum_{k=1}^{(n-1)/2}\frac{n^2}{k^2}\int_0^{\pi/n}t\Big[|f(x+t+2k\pi/n)-f(x)|$$

$$+|f(x+t+(2k-1)\pi/n)-f(x)|\Big]dt$$

$$\leqq A\sum_{k=1}^{(n-1)/2}\frac{n^2}{k^2}\frac{\pi}{n}\left[\int_0^{\pi/n}|f(x+t+2k\pi/n)-f(x)|\,dt\right.$$

$$\left.+\int_0^{\pi/n}|f(x+t+(2k-1)\pi/n)-f(x)|\,dt\right]$$

$$=J_{11}^{21}+J_{11}^{22},$$

say, then by Abel's lemma

$$J_{11}^{21} \leqq A \sum_{k=1}^{(n-1)/2} \frac{n}{k^2} \int_0^{\pi/n} | f(x+t+2k\pi/n) - f(x) | \, dt$$

$$\leqq A \sum_{k=1}^{n} \frac{n}{k^2} \int_{2k\pi/n}^{(2k+1)\pi/n} | f(x+t) - f(x) | \, dt$$

$$\leqq An \sum_{k=1}^{n} \frac{1}{k^3} \int_0^{(2k+1)\pi/n} | f(x+t) - f(x) | \, dt + \frac{1}{n} \int_0^{2\pi} | f(x+t) - f(x) | \, dt$$

$$\leqq A \sum_{k=1}^{n} \frac{1}{k^2} \cdot \frac{n}{k} \int_0^{(2k+1)\pi/n} | f(x+t) - f(x) | \, dt + o(1).$$

For any $\varepsilon$ there is a $\delta$ such that $\frac{1}{\delta} \int_0^{\delta} | f(x+t) - f(x) | \, dt < \varepsilon$ and then for an absolute constant $A$

$$J_{11}^{21} = A \left[ \sum_{k=1}^{(\delta n/\pi - 1)/2} + \sum_{k=(\delta n/\pi - 1)/2}^{n} \right] \frac{1}{k^2} \cdot \frac{n}{k} \int_0^{(2k+1)\pi/n} | f(x+t) - f(x) | \, dt + o(1)$$

$$\leqq A\varepsilon + A/\delta^2 n + o(1) = A\varepsilon + o(1),$$

that is, $\limsup_{n\to\infty} J_{11}^{21} \leqq A\varepsilon$. Since $\varepsilon$ is arbitrary $J_{11}^{21} = o(1)$, and hence $J_{11}^{2} = o(1)$.

We can similary estimate $J_2$, and then the proof of Theorem 6 is complete.

It is easy to see that in Theorem 6 "for all $x$" may be replaced by "for all $x$ in $(a, b)$", and "for all Lebesgue points" by "for all Lebesgue points in $(a, b)$".

## References


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[3] M. Satô: Uniform convergence of Fourier series. III, Proc. Japan Acad., **30** (1954).