166. *Simplification of the Canonical Spectral Representation of a Normal Operator in Hilbert Space and Its Applications*

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(Comm. by K. KUNUGI, M.J.A., Dec. 12, 1955)

If we denote by $K(z)$ the complex resolution of the identity associated with a normal operator N [1] in the abstract Hilbert space $\mathfrak{H}$ which is linear, metric, complete, infinite-dimensional, and separable, and by G the entire complex z-plane, then $N=\int_G z dK(z)$, as is well known.

One of the aims of this note is to outline that if N is an invertible [3] normal operator whose point spectrum is not empty, the above spectral representation of N can be reduced to a line projection-integral, and the other is to outline its applications to few studies on the distribution of the spectrum and the resolvent set of N and to the problem of unitary equivalence of invertible normal operators whose point spectra are not empty. Details of these studies will be, however, shortly published in Memoirs of the Faculty of Education of Kumamoto University.

By the operational calculus we can first prove the following lemmas:

Lemma 1. Let U be a unitary operator in $\mathfrak{H}$; let α and β be arbitrary complex numbers such that $\alpha\overline{\beta}\neq\overline{\alpha}\beta$ and $\overline{\beta}/\beta$ belongs to the resolvent set of U, under the assumption that there exist complex numbers with absolute value 1 in the resolvent set of U; and let H be the operator defined by the relation $H(\beta U-\overline{\beta}I)=\alpha U-\overline{\alpha}I$. Then U and H are permutable, and $(\alpha l-\overline{\alpha})(\beta l-\overline{\beta})^{-1}$ belongs to the point spectrum, to the continuous spectrum, or to the resolvent set of H, according as $l(\neq\overline{\beta}/\beta)$ belongs to the point spectrum, to the continuous spectrum, or to the resolvent set of U; and the converse is also valid. Moreover the characteristic projection of U for a characteristic value l is identical with that of H for the corresponding characteristic value $(\alpha l-\overline{\alpha})(\beta l-\overline{\beta})^{-1}$.

Lemma 2. The preceding lemma holds also in the case where $\overline{\beta}/\beta$ belongs to the continuous spectrum of U.

Remark. In Lemma 1 H is a bounded self-adjoint operator, and in Lemma 2 H is an unbounded self-adjoint operator.

By reference to Lemma 1, we can show the following lemma:

Lemma 3. Let $\mathfrak{A}_\zeta$ be the class of all unitary operators whose resolvent sets contain the point $\zeta=\bar{\beta}/\beta$, $(0<2\arg\bar{\beta}\leqq 2\pi)$, and let $\mathfrak{B}$ be the class of all bounded self-adjoint operators with domain $\mathfrak{H}$. Then there exists a one-to-one correspondence between $\mathfrak{A}_\zeta$ and $\mathfrak{B}$ such that corresponding operators U and H belonging to $\mathfrak{A}_\zeta$ and $\mathfrak{B}$ respectively are related in the form $H(\beta U-\bar{\beta}I)=\alpha U-\bar{\alpha}I$ where α is an arbitrarily given complex number with $\alpha\bar{\beta}\neq\bar{\alpha}\beta$.

By applying Lemma 3 to a well-known theorem [4], we can derive

Lemma 4. Let $\mathfrak{A}_\zeta$ be the class of all unitary operators whose continuous spectra contain the point $\zeta=\bar{\beta}/\beta$, $(0<2\arg\bar{\beta}\leqq 2\pi)$, and $\mathfrak{B}$ the class of all unbounded self-adjoint operators with their respective domains everywhere dense in $\mathfrak{H}$. Then there exists a one-to-one correspondence of the same type as in Lemma 3 between $\mathfrak{A}_\zeta$ and $\mathfrak{B}$.

Definition 1. If the point spectrum and the continuous spectrum of a normal operator N lie on a simple curve Γ in the complex plane, Γ is called the characteristic curve of N, any simple curve with this property being regarded as identical with Γ.

Let $A(N)$ and $B(N)$ be the spectrum and the resolvent set of N respectively, and l an arbitrary complex number. Then $A(N)-l$ and $B(N)-l$ are the spectrum and the resolvent set of $N-lI$ respectively, and $N-lI$ is normal. We may, therefore, assume without loss of generality that N is invertible.

By means of a familiar theorem on factorization and of the preceding lemmas, we can establish the following lemma:

Lemma 5. If N is an invertible bounded (or unbounded) normal operator with point spectrum, N has a unique characteristic curve Γ, and Γ possesses the properties such that (1) Γ is a finite (or an infinite) curve and (2) the absolute value of a point z on Γ is a monotone-increasing or monotone-decreasing function of the argument of z.

Definition 2. If the absolute value of a point z on the characteristic curve Γ of an invertible normal operator with point spectrum is a monotone-increasing function of $\arg(z)$, N is called to belong to the first class $\mathfrak{N}_1$; and if the absolute value of z on Γ is a monotone-decreasing function of $\arg(z)$, N is called to belong to the second class $\mathfrak{N}_2$.

In accordance with Lemma 5, any invertible normal operator with point spectrum belongs either to $\mathfrak{N}_1$ or to $\mathfrak{N}_2$.

On the basis of the preliminaries carried out hitherto, we can establish the following theorem:

Theorem 1. Let N be an invertible (bounded or unbounded) normal operator with point spectrum in $\mathfrak{N}_1$; let Γ be its characteristic curve; and let $E(\mu)$ and $F(\lambda)$ be the resolutions of the identity for the unitary operator U and the positive definite self-adjoint operator H in the polar decomposition $N=UH$ respectively. Then $N=\int_\Gamma zdK(z)$, where $z=\lambda e^{i\mu}$ and $K(z)=E(\mu)F(\lambda)$.

Outline of proof. We can first find that the family of projections $E(\mu)F(\lambda)$ for all $z\in\Gamma$ forms a complex resolution of the identity [2]. We next divide Γ into n segments by points $z_\nu=\lambda_\nu e^{i\mu_\nu}$, $\nu=1,2,\dots,n-1$, and forms the sum $\sum_{\nu=1}^{n} z_{\nu-1}(K(z_\nu)-K(z_{\nu-1}))=\sum_{\nu=1}^{n} z_{\nu-1}(E(\mu_\nu)F(\lambda_\nu)-E(\mu_{\nu-1})F(\lambda_{\nu-1}))$ where the points $z_0=\lambda_0 e^{i\mu_0}$ and $z_n=\lambda_n e^{i\mu_n}$ denote the initial and terminal points of Γ respectively. Thus we can prove that when $n\to\infty$ and each $\mu_\nu-\mu_{\nu-1}\to 0$, the limit of the sum is equal to the projection-integral $\int_G zdK(z)$ over the entire complex plane G.

Remark. We can indicate that if the point spectrum of N is empty, the above theorem is not necessarily valid.

For N, Γ, and $K(z)$ in Theorem 1 we introduce the following definitions:

Definition 3. We denote by $z_1 \ll z_2$ the relation between two points z_1, z_2 on Γ such that $\arg(z_1)<\arg(z_2)$, in consideration of the fact that $|z_1|<|z_2|$ for $\arg(z_1)<\arg(z_2)$.

Definition 4. Let α and z be arbitrary points on Γ. When z tends to α along Γ, we denote by $K(\alpha+0)$ or by $K(\alpha-0)$ the limit of $K(z)$ according as $\alpha\ll z$ or $z\ll\alpha$.

By making use of these definitions we can find the fundamental properties of $K(z)$ on Γ, which are analogous to those of $F(\lambda)$ on the real axis; and applying those properties and the facts that if $N\in\mathfrak{N}_2$, $N^*\in\mathfrak{N}_1$, and the characteristic curves of N and N^* are symmetric with respect to the real axis, we can establish the following assertions:

Theorem 2. If N is an invertible normal operator with point spectrum, then

(1) a necessary and sufficient condition that the linear measure of the continuous spectrum of N be not zero is that the sum of all characteristic projections of N be less than the identity operator I, under the assumption that the point spectrum $\{z_\nu\}$ of N is a finite set or that the set of all limiting points of $\{z_\nu\}$ is at most a denumerably infinite set;

(2) a necessary and sufficient condition that the continuous spectrum of N consist only of limiting points $\{\alpha_\mu\}$ of characteristic values of N is that the sum of all characteristic projections of N be identical with I, under the assumption that the set $\{\alpha_\mu\}$ is at most a denumerably infinite set.

Theorem 3. Let N be an invertible normal operator whose point spectrum $\{z_\nu\}$ is denumerably infinite and not everywhere dense on the corresponding characteristic curve Γ; and let the sum of all characteristic projections of N be identical with I. Then any point belonging neither to $\{z_\nu\}$ nor to its limiting points is in the resolvent set of N.

Theorem 4. Let N be an invertible normal operator, $\{z_\nu\}$ its point spectrum, and $\{P_\nu\}$ the set of the corresponding characteristic projections of N. Then a necessary and sufficient condition that N be expressible in the form $N=\sum_\nu z_\nu P_\nu$ is that $\sum_\nu P_\nu=I$.

Definition 5. Let N, Γ, and $K(z)$ be those in Theorem 1 respectively; let $\rho(z)$ denote $\|K(z)f\|^2$ for $z \in \Gamma$ and $f \in \mathfrak{D}(N)$; let $\mathfrak{L}_2(f)$ denote the class of all complex-valued ρ-measurable functions $F(z)$ such that $|F(z)|^2$ is ρ-integrable along Γ, two functions $F(z)$ and $G(z)$ being regarded as identical if and only if they are ρ-equivalent; and let $\mathfrak{M}(f)$ denote the set of all elements $f^*=F(N)f$ where $F(z)$ belongs to $\mathfrak{L}_2(f)$.

Making use of $\mathfrak{M}(f)$ defined above, by methods analogous to those used in the proofs of well-known theorems [5] concerning the unitary equivalence of self-adjoint operators we can derive two theorems concerning the unitary equivalence of invertible normal operators with point spectra and can find that the two theorems are exactly analogous to the theorems in the case of self-adjoint operators.

Applying the line projection-integral stated in Theorem 1 and one of the two theorems, we can prove the following theorem of a new form:

Theorem 5. Let $N_i, i=1,2$, be invertible normal operators with point spectra in $\mathfrak{N}_1$, Γ_i the characteristic curve of N_i, $K_i(z)$ the complex resolution of the identity for N_i, $\mathfrak{C}_i$ the set of all characteristic elements of N_i, $\mathfrak{M}_i$ the closed linear manifold determined by $\mathfrak{C}_i+\mathfrak{O}$, and $\mathfrak{N}^{(i)}$ the orthogonal complement of $\mathfrak{M}_i$. Then there exist orthonormal sets $\{\psi_\nu^{(i)}\}$, $i=1,2$, such that (1) $\mathfrak{N}^{(i)}=\mathfrak{M}(\psi_1^{(i)})\oplus\mathfrak{M}(\psi_2^{(i)})\oplus\ldots$, (2) $\|K_i(z)\psi_\nu^{(i)}\|^2$ is a continuous function of $z \in \Gamma_i$, and (3) $\psi_1^{(i)}\succ\psi_2^{(i)}\succ\psi_3^{(i)}\succ\cdots$; and for the unitary equivalence of N_1 and N_2 it is necessary and sufficient that

(I) if the sum of all characteristic projections of N_1 and that of all characteristic projections of N_2 are both identical with I, N_1, and N_2 have the same point spectrum (inclusive of the multiplicities of characteristic values) and continuous spectrum;

(II) if the sum of all characteristic projections of N_1 and that of all characteristic projections of N_2 are both less than I, then, in addition to the above property concerning the spectra, N_1 and N_2 enjoy such properties that the orthonormal sets $\{\psi_\nu^{(i)}\}$, $i=1,2$, have the same number of elements and satisfy the relations $\|K_1(z)\psi_\nu^{(1)}\|^2 \sim \|K_2(z)\psi_\nu^{(2)}\|^2$, $\nu=1,2,\dots$, for every z belonging to the common continuous spectrum of N_1 and N_2. Moreover, if N_i, $i=1,2$, belong to $\mathfrak{N}_2$, a necessary and sufficient condition that N_1 and N_2 be unitarily equivalent is that the same condition as above be fulfilled by N_i^*, $i=1,2$.

Remark. For an invertible normal operator N with empty point spectrum, we can also verify that Theorem 1 is valid under the assumption that the self-adjoint operator H in the polar decomposition of N is expressible in the form $H=(\alpha U-\bar{\alpha}I)(\beta U-\bar{\beta}I)^{-1}$ where U is the unitary operator in that decomposition. Moreover, if two invertible normal operators N_1, N_2 with empty point spectra satisfy the above conditions $H_i=(\alpha_i U_i-\bar{\alpha}_i I)(\beta_i U_i-\bar{\beta}_i I)^{-1}$, $i=1, 2$, respectively, we can establish the following assertions:

1° a necessary and sufficient condition for the unitary equivalence of N_1 and N_2 in the first class $\mathfrak{N}_1$ is that N_1 and N_2 have the same continuous spectrum Δ and the sets $\{\psi_\nu^{(i)}\}$, $i=1,2$, which have the same number of elements, satisfy the relations $\|K_1(z)\psi_\nu^{(1)}\|^2 \sim \|K_2(z)\psi_\nu^{(2)}\|^2$, $\nu=1,2,3,\dots$, for every $z \in \Delta$;

2° for N_1 and N_2 in the second class $\mathfrak{N}_2$, the above assertion 1° holds for Γ', $K_i'(z)$ and Δ' associated with N_i^*, $i=1,2$.

References

[1] Béla v. Sz. Nagy: Spektraldarstellung linearer Transformationen des Hilbertschen Raumes, Berlin, 18, 33 (1947).

[2] ——: Cf. p. 19.

[3] Paul R. Halmos: Introduction to Hilbert space and the theory of spectral multiplicity, New York, 37 (1951).

[4] Marshall H. Stone: Linear transformations in Hilbert space and their applications to analysis, New York, 304–307 (1932).

[5] ——: Cf. pp. 262–266, 272–275.