161. On Singular Cross Sections

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1. The theory of obstructions to extentions and homotopies developed by Paul Olum in the paper [3] is generalized to that of cross sections of fibre spaces. Our purpose of the present paper is to give the definition of singular cross sections and their obstruction cocycles, and to state some theorems concerning extentions of cross sections without the proof.

2. Let (X, p, B) be a pseudo fibre space such that the total space X is arcwise connected (p. 63 of [1]; p. 443 of [4]). The projection $p: X \rightarrow B$ induces a singular mapping $p: S(X) \rightarrow S(B)$ of the singular complex of X into the singular complex of B ((7.1) of [3]). Let S' be a subcomplex of $S(B)$. *A singular cross section φ over S' is defined to be a singular mapping $\varphi: S' \rightarrow S(X)$ which satisfies the condition:*

(1) $$\bar{p}\varphi = 1.$$

Let (X, p, K) be a pseudo fibre space such that X is arcwise connected and K is a CW-complex [6]. Let L be a subcomplex of K and let $\varphi: \bar{K}^n = K^n \smile L \rightarrow X$ be a cross section. The map φ induces a singular cross section $\bar{\varphi}: S(L) \smile S^n(K) \rightarrow S(X)$, where $S^n(K)$ is the n-section of $S(K)$.

Theorem 1. A cross section $\varphi: \bar{K}^n \rightarrow X$ is extended over $\bar{K}^{n+1}$ if and only if the singular cross section $\bar{\varphi}: S(L) \smile S^n(K) \rightarrow S(X)$ induced by φ is extended over $S(L) \smile S^{n+1}(K)$.

3. Let $\bar{B} = B \times I$ be the Cartesian product of a given space B and the unit interval I. We identify B with the subspace $B \times 0$; then, $S(B)$ is a subcomplex $S_0 = S(\bar{B} \times 0)$ of $S(\bar{B})$. Let $S_1 = S(B \times 1)$. Define maps $\rho_B: \bar{B} \rightarrow B$, $\sigma_B: \bar{B} \rightarrow I$ by

$$\rho_B(b, t) = b, \qquad \sigma_B(b, t) = t.$$

The map ρ_B induces a singular mapping $\bar{\rho}_B: S(\bar{B}) \rightarrow S(B)$.

Let (X, p, B) be as in §2. The spaces $\bar{X} = X \times I$, $\bar{B} = B \times I$ together with a map $q: \bar{X} \rightarrow \bar{B}$ defined by $q(x, t) = (p(x), t)$ form a pseudo fibre space $(\bar{X}, q, \bar{B})$. Let $\varphi_0, \varphi_1: S(\bar{B}) \rightarrow S(X)$ be singular cross sections such that φ_0 agrees with φ_1 over a subcomplex S' of $S(B)$. Let $\bar{S}'$ be a subcomplex of $S(\bar{B})$ defined as in §4 of [3]. *If there is a singular cross section $\Phi: S_0 \smile \bar{S}' \smile S^{n+1}(B) \smile S_1 \rightarrow S(\bar{X})$ such that*

$$\Phi(\bar{u})=\begin{cases}(\varphi_0(\bar{u}), 0) & \text{if } \bar{u} \in S_0\\ (\varphi_1(\bar{\rho}_B\bar{u}), 1) & \text{if } \bar{u} \in S_1\\ (\varphi_0(\bar{\rho}_B\bar{u}), \sigma_B\bar{u}) & \text{if } \bar{u} \in \bar{S}',\end{cases}$$

we say that the map φ_0 is n-homotopic[1] *to φ_1 rel. S'.*

Theorem 2. Let (X, p, K) be as in § 2. Let L be a subcomplex of K and let $\varphi_0, \varphi_1: K \to X$ be cross sections such that $\varphi_0 | L = \varphi_1 | L$. Then, φ_0 is n-homotopic[2] *to φ_1 rel. L if and only if the singular cross section $\bar{\varphi}_0: S(K) \to S(X)$ induced by φ_0 is n-homotopic rel. $S(L)$ to the singular cross section $\bar{\varphi}_1: S(X) \to S(X)$ induced by φ_1.*

4. Let (X, p, B) be as above. Let S' be a subcomplex of $S(B)$. A singular cross section $\varphi: S' \to S(X)$ is extended over $S' \smile S^0(B)$.

Theorem 3. A singular cross section $\varphi: S' \smile S^0(B) \to S(X)$ is extended over $S' \smile S^1(B)$ if and only if the fibre F_b over a point $b \in B$ is arcwise connected.

Let $\varphi: S^n(B) \to S(X)$, $n \geq 2$, be a singular cross section. The map φ induces a homomorphism $\theta_k: \pi_k(B, b) \to \pi_k(X, \varphi(b))$, $b \in B$, for each integer k such that $1 \leq k \leq n-1$. Let $p_k^*: \pi_k(X, \varphi(b)) \to \pi_k(B, b)$ be a homomorphism induced by $p: X \to B$. Then, θ_k satisfies the condition:

$$(2) \qquad p_k^* \theta_k = 1, \qquad 1 \leq k \leq n-1.$$

In general, a homomorphism $\theta: \pi_k(B, b) \to \pi_k(X, x)$, $p(x) = b$, such that $p_k^* \theta = 1$, will be called to be special. Let $\Delta_k: \pi_k(B, b) \to \pi_{k-1}(F_b, x)$, $x \in F_b = p^{-1}(b)$, be the boundary homomorphism. If there exists a singular cross section $\varphi: S^n(B) \to S(X)$, $n \geq 2$, the boundary homomorphism Δ_k is trivial, for each k such that $2 \leq k \leq n$.

Theorem 4. Let S' be a subcomplex of $S(B)$. In order that a singular cross section $\varphi: S' \smile S^0(B) \to S(X)$, $n \geq 2$, is extended over $S' \smile S^n(B)$, it is necessary and sufficient that there exists a special homomorphism $\theta_{n-1}: \pi_{n-1}(B, b) \to \pi_{n-1}(X, \varphi(b))$, $b \in S' \smile S^0(B)$, such that φ is consistent[3] *with θ_{n-1} and the boundary homomorphism $\Delta_n: \pi_n(B, b) \to \pi_{n-1}(F_b, \varphi(b))$ is trivial.*

Theorem 5. Let S' be a connected[4] *non-vacuous subcomplex of $S(B)$. A singular cross section $\varphi': S' \to S(X)$ is extended over $S' \smile S^2(B)$ if and only if there exists a special homomorphism $\theta: \pi_1(B, b) \to \pi_1(X, \varphi(b))$, $b \in S'$, such that φ is consistent with θ, the fibre F_b over b is arcwise connected, and the boundary homomorphism Δ_2 is trivial.*

Theorem 6. In order that there exists a singular cross section $\varphi: S^2(B) \to S(X)$, it is necessary and sufficient that there exists a

1) For the definition of "free n-homotopic", refer to (7.4) of [3].

2) See § 32.5 of [5].

3) For the definition, refer to § 9 of [3].

4) For the definition, see (9.4) of [3].

special homomorphism $\theta:\pi_1(B,b)\to\pi_1(X,x)$, $b\in B$, $x\in p^{-1}(b)=F_b$, *the fibre* F_b *is arcwise connected and the boundary homomorphism* Δ_2 *is trivial.*

Theorem 7. Let S' *be a connected non-vacuous subcomplex of* $S(B)$, *and let* $\varphi_0,\varphi_1: S'\smile S^2(B)\to S(X)$ *be two singular cross sections such that* $\varphi_0|S'=\varphi_1|S'$. *Then,* $\varphi_0\simeq\varphi_1$ *dim* 1 *rel.* S' *if and only if* φ_0,φ_1 *induce a same homomorphism* $\theta:\pi_1(B,b)\to\pi_1(X,\varphi_0(b))$, $b\in S'$.

Theorem 8. Let $\varphi_0,\varphi_1: S^2(B)\to S(X)$ *be two singular cross sections, and let* φ_0,φ_1 *induce the homomorphisms* $\theta_0:\pi_1(B,b)\to\pi_1(X,\varphi_0(b))$, $\theta_1:\pi_1(B,b)\to\pi_1(X,\varphi_1(b))$ *respectively, where* b *is a fixed point of* B. *Let* ω *be a class of paths from* $\varphi_0(b)$ *to* $\varphi_1(b)$ *in* $F_b=p^{-1}(b)$. *Then,* $\varphi_0\simeq\varphi_1$ *dim* 1 *(*x_0 *thru* ω*)*[5] *if and only if* $\theta_0=\omega\circ\theta_1$.

5. Let (X,p,B) be as above, x be a point of X and F_x be the fibre over $p(x)$. Then, the collection of the n-th homotopy groups, $\pi_n=\{\pi_n(F_x,x)\,|\,x\in X\}$, form a system of local groups over X (Theorem 1 of [2]). Let $\varphi: S^2(B)\to S(X)$ be a singular cross section. Then, a subcollection, $\Pi_n(\varphi)=\{\pi_n(F_x,x)\,|\,x=\varphi(b), b\in B\}$ of Π_n form a system of local groups over B.

Let S' be a subcomplex of $S(B)$ and let $\varphi: S'\smile S^n(B)\to S(X)$, $n\geq 1$, be a singular cross section. In the sequel of this section, we assume that the fundamental group $\pi_1(F_x,x)$ is abelian, if $n=1$. In this case, since F_x is 1-simple, $\Pi_1(\varphi)=\{\pi_1(F_x,x)\,|\,x=\varphi(b), b\in B\}$ form a system local groups over B. Let $\sigma_{n+1}=a_0a_1\ldots a_{n+1}$ be an ordered Euclidean $(n+1)$-simplex, and let $u_{n+1}:\sigma_{n+1}\to B$ be an $(n+1)$-simplex of $S(B)$ with leading vertex $b=u_{n+1}(a_0)$. Define a map $f:\dot\sigma_{n+1}\to B$ as in § 11 of [3]. The map $\varphi f:(\dot\sigma_{n+1},a_0)\to(X,\varphi(b))$ determines an element α of $\pi_n(X,\varphi(b))$. Let $\chi:(E^n_+,S^{n-1})\to(\dot\sigma_{n+1},a_0)$ be an order preserving map. Since $p\varphi f\chi:(E^n_+,S^{n-1})\to(B,b)$ is inessential, there exists a map $g:(S^n,S^{n-1})\to(X,\varphi(b))$ such that $g|E^n_+=\varphi f\chi$ and $g(E^n_-)\subseteq p^{-1}(b)=F_b$. The map $g:(E^n_-,S^{n-1})\to(F_b,\varphi(b))$ determines an element $\alpha(\varphi,u_{n+1})\in\pi_n(F_b,\varphi(b))$ which depends only on φ and u_{n+1}. Now, we define a cochain $C^{n+1}(\varphi)\in C^{n+1}(S(B),S';\Pi_n(\varphi))$ by setting, for $u_{n+1}\in S(B)$,

$$C^{n+1}(\varphi)(u_{n+1})=\alpha(\varphi,u_{n+1}),$$

where $C^{n+1}(S(B),S';\Pi_n(\varphi))$ is the $(n+1)$-th cochain group of $S(B)$ mod S' with the system of local groups $\Pi_n(\varphi)$ as coefficient domain. By slightly modifying the proof of Lemma 11.5 of [3], one can prove that $C^{n+1}(\varphi)$ is a cocycle. Then, we say that $C^{n+1}(\varphi)$ is the obstruction cocycle of φ and we denote by $h^{n+1}(\varphi)$ the element of $H^{n+1}(S(B),S';\Pi_n(\varphi))$ determined by $C^{n+1}(\varphi)$.

Theorem 9. A singular cross section $\varphi: S'\smile S^n(B)\to S(X)$, $n\geq 1$, *is extended over* $S'\smile S^{n+1}(B)$ *if and only if* $C^{n+1}(\varphi)=0$.

5) See (7.3) and Theorem (9.11) of [3].

Theorem 10. Given a singular cross section $\varphi : S' \smile S^n(B) \rightarrow S(X)$, $n \geq 1$, *there exists a singular cross section* $\psi : S' \smile S^{n+2}(B) \rightarrow S(X)$ *which agrees with* φ *on* $S' \smile S^{n+1}(B)$ *if and only if* $h^{n+1}(\varphi) = 0$.

Another results corresponding to that given in [3] are easily formulated in our case and are all omitted.

References

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