159. *Cohomology of the Three-fold Symmetric Products of Spheres*

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(Comm. by K. KUNUGI, M.J.A., Dec. 12, 1955)

§1. **Introduction.** Let K be a space, and denote by $K^n = K \times K \times \cdots \times K$ the n-fold Cartesian product of K. Then we may regard the symmetric group $\mathfrak{S}_n$ of degree n as a transformation group acting on K^n in a natural fashion as follows: For any $\gamma \in \mathfrak{S}_n$ and $(x_1, x_2, \ldots, x_n) \in K^n$, we set $\gamma(x_1, x_2, \ldots, x_n) = (x_{\gamma(1)}, x_{\gamma(2)}, \ldots, x_{\gamma(n)})$. The orbit space over K^n relative to $\mathfrak{S}_n$ will be called *the n-fold symmetric product* of K.

In the present paper, we shall determine the cohomology of the 3-fold symmetric product $S^n * S^n * S^n$ of an n-sphere $S^n (n \geqq 1)$, by making use of the results and arguments in the previous paper.[1] Full details will appear in the Journal of the Institute of Polytechnics, Osaka City University.

§2. **Methods for calculations.** Let

$$T, S : S^n \times S^n \times S^n \to S^n \times S^n \times S^n$$

be transformations given by

$$T(x_1, x_2, x_3) = (x_2, x_3, x_1),$$
$$S(x_1, x_2, x_3) = (x_2, x_1, x_3), \qquad (x_1, x_2, x_3 \in S^n)$$

respectively. Then the orbit space over $S^n \times S^n \times S^n$ relative to T is the 3-fold cyclic product ϑ_{n3} of S^n,[2] whose cohomology has determined in CP. Since $TS = ST^2$, $T^2S = ST$, the transformation $S: S^n \times S^n \times S^n \to S^n \times S^n \times S^n$ induces a transformation $\bar{S} : \vartheta_{n3} \to \vartheta_{n3}$ such that $\pi S = \bar{S}\pi$, where $\pi : S^n \times S^n \times S^n \to \vartheta_{n3}$ is the natural projection. Then $\bar{S}$ is the transformation of period 2, and the orbit space over ϑ_{n3} relative to $\bar{S}$ is the symmetric product $S^n * S^n * S^n$. Note that the set of fixed points under $\bar{S}$ is homeomorphic with $S^n \times S^n$. We shall now apply the theory in §1 of CP to the complex ϑ_{n3} with the transformation $\bar{S}$. Then we obtain the results stated in the following.

§3. **The mod 2 cohomology.** The cohomology groups $H^r(S^n * S^n * S^n; Z_2)$ with coefficients in Z_2 are as follows:[3]

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1) Nakaoka, M.,: Cohomology of the p-fold cyclic products, Proc. Japan Acad., **31** (1955). We refer to this paper as CP.

2) This is the notation used in Liao, S. D.,: On the topology of cyclic products of spheres, Trans. Amer. Math. Soc., **77** (1954).

3) We shall write Z and Z_p respectively for the group of integers and the group of integers mod p.

(3.1) *$H^r(S^n * S^n * S^n; Z_2) \approx Z_2$ for $r=0, n, n+2 \leqq r \leqq 2n$ and $2n+2 \leqq r \leqq 3n$; $=0$ for other r.*

Let $b^{\#}_{n+i}$ ($i=0, 2\leqq i \leqq n, n+2 \leqq i \leqq 2n$) be the generator of $H^{n+i}(S^n * S^n * S^n; Z_2)$, and let Sq^i and $\smile$ denote the Steenrod square and the cup product respectively. Then we have

(3.2) i) $Sq^i(b^{\#}_n) = b^{\#}_{n+i}$ *for* $2 \leqq i \leqq n$.

ii) *If* $k=1, 2$, *and* $1 \leqq j \leqq n-1$, *then*

$$Sq^i(b^{\#}_{kn+j+1}) = {}_jC_i\, b^{\#}_{kn+i+j+1} \quad \text{if } i+j \leqq n-1,$$
$$= 0 \quad \text{if } i+j > n-1,$$

where ${}_jC_i$ denotes the binomial coefficient with the usual conventions.

(3.3) i) $b^{\#}_n \smile b^{\#}_{n+i} = b^{\#}_{2n+i}$ *for* $2 \leqq i \leqq n$.

ii) $b^{\#}_{n+i} \smile b^{\#}_{n+j} = 0$ *for* $2 \leqq i, j \leqq n$.

It is to be noticed that:

(3.4) *Let $\pi_0 : S^n \times (S^n * S^n) \to S^n * S^n * S^n$ be the natural projection, then the homomorphism $\pi_0^* : H^r(S^n * S^n * S^n; Z_2) \to H^r(S^n \times (S^n * S^n); Z_2)$ induced by π_0 is isomorphic into for any r.*

§4. **The mod 3 cohomology.** Using the notations in CP, we can take as a base for $H^*(\vartheta_{n3}; Z_3)$ the following:

$$1^{\#},\ g^{\#}_n(1),\ g^{\#}_{2n}(1, 2), \text{ and } a^{\#}_{n+s}\ (2 \leqq s \leqq 2n).$$

Let $\bar{S} : C^r(\vartheta_{n3}; Z_3) \to C^r(\vartheta_{n3}; Z_3)$ be the cochain map induced by $\bar{S}$, and $\bar{\pi} : C^r(S^n * S^n * S^n; Z_3) \to C^r(\vartheta_{n3}; Z_3)$ the cochain map induced by the natural projection. Then we can define a cochain map $\phi : C^r(\vartheta_{n3}; Z_3) \to C^r(S^n * S^n * S^n; Z_3)$ by $\bar{\pi}\phi = 1 + \bar{S}$. Let $\phi^* : H^r(\vartheta_{n3}; Z_3) \to H^r(S^n * S^n * S^n; Z_3)$ be the homomorphism induced by ϕ. Write $\bar{g}^{\#}_n = \phi^* g^{\#}_n(1)$, $\bar{g}^{\#}_{2n} = \phi^* g^{\#}_{2n}(1, 2)$, $\bar{a}^{\#}_{n+s} = \phi^* a^{\#}_{n+s}$. Then we have

(4.1) *As a base for the vector space $H^*(S^n * S^n * S^n; Z_3)$, we can take the following:*

$1^{\#}$, $\bar{g}^{\#}_n$, $\bar{g}^{\#}_{2n}$ (n: even), $\bar{a}^{\#}_{n+4\alpha+1}$ ($1 \leqq \alpha \leqq [(2n-1)/4]$), and $\bar{a}^{\#}_{n+4\alpha}$ ($1 \leqq \alpha \leqq [n/2]$),[4] where $[k]$ denotes the greatest integer $\leqq k$.

Denote by $\Delta_3 : H^r(S^n * S^n * S^n; Z_3) \to H^{r+1}(S^n * S^n * S^n; Z_3)$, and $\mathcal{P}^i : H^r(S^n * S^n * S^n; Z_3) \to H^{r+4i}(S^n * S^n * S^n; Z_3)$ the Bockstein homomorphism and the reduced power respectively. Then we have

(4.2) i) $\mathcal{P}^i \bar{g}^{\#}_n = (-1)^{i+1} \bar{a}^{\#}_{n+4i}$ ($i \neq 0$),

ii) $\mathcal{P}^i \bar{g}^{\#}_{2n} = 0$ ($i \neq 0$),

iii) $\mathcal{P}^i \bar{a}^{\#}_{n+4\alpha+1} = {}_{2\alpha}C_i \bar{a}^{\#}_{n+4(\alpha+i)+1}$,

iv) $\mathcal{P}^i \bar{a}^{\#}_{n+4\alpha} = {}_{2\alpha-1}C_i \bar{a}^{\#}_{n+4(\alpha+i)}$.

(4.3) i) $\Delta_3 \bar{g}^{\#}_n = 0$, ii) $\Delta_3 \bar{g}^{\#}_{2n} = 0$, iii) $\Delta_3 \bar{a}^{\#}_{n+4\alpha+1} = 0$,

iv) $\Delta_3 \bar{a}^{\#}_{n+4\alpha} = \bar{a}^{\#}_{n+4\alpha+1}$.

(4.4) i) $\bar{g}^{\#}_n \smile \bar{g}^{\#}_n = (-1)^{n/2+1} \bar{a}^{\#}_{3n}$ (n: *even*),

$= 0$ (n: *odd*).

ii) $\bar{g}^{\#}_n \smile \bar{a}^{\#}_{n+4\alpha+\varepsilon} = 0$ ($\varepsilon = 0, 1$),

4) The lower suffix denotes the dimension.

$$\bar{a}^{\#}_{n+4\alpha+\varepsilon} \smile \bar{a}^{\#}_{n+4\beta+\varepsilon'}=0 \qquad (\varepsilon, \varepsilon'=0, 1).$$

§5. Integral homology groups. Let A be an abelian group. Then we denote by $C(A, p)$ the p-primary component of A, and $C(A, \infty)$ the free component of A. For the integral homology group $H_r(S^n * S^n * S^n; Z)$, we have the following results:

(5.1) i) *$C(H_r(S^n*S^n*S^n; Z), \infty) \approx Z$ for $r=0, n, 2n$ with even n, $3n$ with even n; $=0$ for other r.*

ii) *$C(H_r(S^n*S^n*S^n; Z), 2) \approx Z_2$ for $r=jn+2k$ with $1 \leqq k \leqq [(n-1)/2]$ and $j=1, 2$; $=0$ for other r.*

iii) *$C(H_r(S^n*S^n*S^n; Z), 3) \approx Z_3$ for $r=n+4k$ with $1 \leqq k \leqq [(2n-1)/4]$; $=0$ for other r.*

iv) *$C(H_r(S^n*S^n*S^n; Z), q)=0$ for odd prime $q \neq 3$ and any r.*

Finally we can verify

(5.2) *The 3-fold symmetric product of an n-sphere and the Eilenberg-MacLane complex $K(Z, n)$ are of the same $(n+4)$-homotopy type.*