28. *A Remark on the Ranged Space*

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1. In this paper, we shall change a given system of neighbourhoods which, in general, does not satisfy the axiom (C) of Hausdorff:

(C) let $v(p)$ be an arbitrary neighbourhood of p and let q be an arbitrary point of $v(p)$, there exists a neighbourhood $v(q)$ such as $v(q)\subseteq v(p)$,

in order to make a new system satisfy the axiom (C), and to make the new system be equivalent to the old one provided the latter satisfies the axiom (C). Then we apply this fact to the ranged space[1] and show that the axiom (C) is needless to define the ranged space.

Definition 1. Consider a space R where the topology T is given by a system $\{v\}$ of neighbourhood satisfying the axiom (A); $p\in v(p)$, and $\{v(p)\}\neq\phi$ for any point p of R. When a set $A\subseteq R$ is given, we shall denote by $\tilde{A}$ the set of all points p which have the following property:

When an arbitrary neighbourhood $v(p)$ of p is given, there exist a point $a\in A$ and a neighbourhood $v(a)$ such as $v(a)\subseteq v(p)$. And we shall denote a new topology, which has $\tilde{A}$ as the closure of A, by T^*. Then it is clear that $\bar{A}\supseteq\tilde{A}$, where $\bar{A}$ means the closure of A with respect to the old topology T.

Theorem 1. *The topology T^* has the following properties:*

(I) $$A\subseteq\tilde{A},$$

(II) $$\textit{if } A\subseteq B, \textit{ then } \tilde{A}\subseteq\tilde{B},$$

(III) $$\tilde{\phi}=\phi,$$

(IV) $$\tilde{\tilde{A}}\subseteq\tilde{A}.$$

Proof. (I) is evident. Let $A\subseteq B$ and let p and $v(p)$ be an arbitrary point $\tilde{A}$ and an arbitrary neighbourhood of p respectively, there exist a point $a\in A$ and a neighbourhood $v(a)$ such as $v(a)\subseteq v(p)$. From $A\subseteq B$ it follows that $a\in B$, therefore $p\in\tilde{B}$. (III) follows from the fact that $\bar{\phi}=\phi$ and T^* is stronger than T. Let p and $v(p)$ be

1) K. Kunugi: Sur les espaces complets et régulièrement complets. I–II, Proc. Japan Acad., **30** (1954).

an arbitrary point of $\tilde{\tilde{A}}$ and an arbitrary neighbourhood of p respectively, there exist a point $q \in \tilde{A}$ and a neighbourhood $v(q)$ such as $v(q) \subseteq v(p)$. Since $q \in \tilde{A}$, there exist a point $a \in A$ and a neighbourhood $v(a)$ such as $v(a) \subseteq v(q) (\subseteq v(p))$. Therefore $p \in \tilde{A}$.

Definition 2. If a neighbourhood $v(p)$ satisfies the axiom (A), then it satisfies a following axiom (C′) which is weaker condition than the axiom (C) of Hausdorff:

(C′) *there exist a point* $q \in v(p)$ *and a neighbourhood* $v(q)$ *such as* $v(q) \subseteq v(p)$.

We shall call the set of all points q having the property (C′), the kernel of $v(p)$ and denote it by $K[v(p)]$.

Then the space R having the topology T^* is the ordinary space of neighbourhoods having the system $\{K[v]\}$ as its system of neighbourhoods. We notice that there is Example[2] that T^* is strictly stronger than T.

Theorem 2. *If the system* $\{v\}$ *satisfies the axiom* (*A*) *and if* $\{v(p)\} \neq \phi$ *for any point* $p \in R$, *then the system* $\{K[v]\}$ *of kernels has the following properties*:

(I*) $p \in K[v(p)]$ *for any* $K[v(p)]$,

(II*) *if* $v_1(p_1) \subseteq v_2(p_2)$, *then* $K[v_1(p_1)] \subseteq K[v_2(p_2)]$ (*monotone*),

(III*) $\{K[v(p)]\} \neq \phi$ *for any point* $p \in R$,

(IV*) *for any point* $q \in K[v(p)]$, *there exists* $v(q)$ *such as* $K[v(q)] \subseteq K[v(p)]$ (*the axiom* (*C*)).

Proof. (I*) and (III*) are evident. If $q \in K[v_1(p_1)]$ and if $v_1(p_1) \subseteq v_2(p_2)$, then there exists $v(q)$ such as $v(q) \subseteq v_1(p_1)$ $(\subseteq v_2(p_2))$, therefore $q \in K[v_2(p_2)]$, i.e. we have (II*). If $q \in K[v(p)]$, then there exists a neighbourhood $v(q)$ such as $v(q) \subseteq v(p)$. By virtue of (II*), it follows that $K[v(q)] \subseteq K[v(p)]$, i.e. we have (IV*).

Theorem 3. *If* $\{u\}$ *and* $\{v\}$ *are equivalent, then* $\{K[u]\}$ *and* $\{K[v]\}$ *are equivalent.*

Proof. Let $K[u(p)]$ be an arbitrary kernel of the system $\{K[u]\}$, there exists $v(p)$ such as $v(p) \subseteq u(p)$. By virtue of monotone of kernels, we have $K[v(p)] \subseteq K[u(p)]$. Exchange $\{u\}$ for $\{v\}$, then it follows that there exists $K[u(p)]$ such as $K[u(p)] \subseteq K[v(p)]$ for any kernel $K[v(p)]$.

Theorem 4. *If the system* $\{v(p)\}$ *satisfies the axiom* (*B*) *of Hausdorff at a point* p:

(B) let $v_1(p)$ and $v_2(p)$ be arbitrary two neighbourhoods of p, there exists a neighbourhood $v_3(p)$ such as $v_3(p) \subseteq (v_1(p) \cap v_2(p))$, *then the system* $\{K[v(p)]\}$ *also satisfies the axiom* (*B*) *at the same point* p.

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2) See Example at the end of this paper.

Proof. It is clear by virtue of monotone of kernels.

Theorem 5. *If the system $\{v\}$ is equivalent to a system $\{u\}$ satisfying the axiom (C) of Hausdorff, then $\{v\}$ and $\{K[v]\}$ are equivalent.*

Proof. By virtue of Theorem 3, it is enough to prove that $\{u\}$ and $\{K[u]\}$ are equivalent. If $\{u\}$ satisfies the axiom (C), then $u(p)=K[u(p)]$. And so we have this theorem.

Theorem 6. *The depth*[3] $\omega^*(R, p)$ *defined by the system* $\{K[v]\}$ *is well defined by the given topology T.*

Proof. This follows from Theorem 3 and from the fact that the depth $\omega^*(R, p)$ is well defined[4] by the topology T^*.

Theorem 7. *Denote the depths defined by $\{v\}$ or by $\{K[v]\}$ respectively by $\omega(R, p)$ or by $\omega^*(R, p)$. Then $\omega^*(R, p)\geq\omega(R, p)$.*

Proof. If $\{v\}$ satisfies the axiom (B) at the point p, then $\{K[v]\}$ also satisfies (B) (Theorem 4), and so $\omega^*(R, p)\neq 0$. In the case when $\omega(R, p)$ is actually infinite, then, by virtue of monotone of kernels, we have that $\omega^*(R, p)$ is also actually infinite. In the case when $\omega(R, p)$ is not actually infinite, consider the sequence of neighbourhoods of the point p:

(1) $$v_0(p)\supseteq v_1(p)\supseteq\cdots\supseteq v_\alpha(p)\supseteq\cdots,\ 0\leq\alpha<\beta,$$

where β is an ordinal number.

By virtue of monotone of kernels, we have

(2) $$K[v_0(p)]\supseteq K[v_1(p)]\supseteq\cdots K\supseteq[v_\alpha(p)]\supseteq\cdots,\ 0\leq\alpha<\beta.$$

If the sequence (1) is not maximal, then, by virtue of monotone of kernels, it follows that the sequence (2) is not maximal. Therefore, $\omega^*(R, p)\geq\omega(R, p)$.

2. Definition 3. Let ω_ν^* be an ordinal limit inaccessible number such as $\omega_\nu^*\leq\omega^*(R)$. We shall call that a ranged space R is *strongly complete* if $\bigcap_\alpha K[v_\alpha(p_\alpha)]$ is non empty, for any fundamental sequence $\{K[v_\alpha(p_\alpha)]\}$, $0\leq\alpha<\omega_\mu^*(\leq\omega_\nu^*)$ of kernels (this means that the sequence $\{K[v_\alpha(p_\alpha)]\}$ is a fundamental sequence of neighbourhoods with respect to T^*).

Lemma. *If $\alpha<\omega^*(R)$ and if*

(1) $$K[v_0(p_0)]\supseteq K[v_1(p_1)]\supseteq\cdots\supseteq K[v_\beta(p_\beta)]\supseteq\cdots,\ v_\beta(p_\beta)\in\mathfrak{B}_{\gamma_\beta},^{5)}$$
$$\gamma_0<\gamma_1<\gamma_2<\cdots<\gamma_\beta<\cdots,\quad \beta<\alpha,$$

then $\bigcap_\beta K[v_\beta(p_\beta)]$ is open with respect to T^*.

Proof. As the kernels satisfy the axiom (C), this theorem is true for $\alpha=1$. Suppose that the theorem has been proved already for $\alpha<\lambda(<\omega^*(R))$.

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3) Cf. 1).

4) Cf. 1).

5) This means that $v_\beta(p_\beta)$ is a neighbourhood of rank γ_β (cf. 1)).

In the case λ is a limit number. Let p be an arbitrary point of $\bigcap_{\beta<\lambda} K[v_\beta(p_\beta)]$. By virtue of the hypothesis that $\bigcap_{\beta<\alpha} K[v_\beta(p_\beta)]$, $\alpha<\lambda$, is open with respect to T^* and from the definition of the ranged space, we can show that there exists a sequence of kernels as follows:

$$(2) \qquad K[v_0(p)]\supseteq K[v_1(p)]\supseteq\cdots\supseteq K[v_\beta(p)]\supseteq\cdots,$$

where $\quad K[v_\beta(p)]\subseteq K[v_\beta(p_\beta)],\ v_\beta(p)\in\mathfrak{V}_{\gamma_\beta},\ \beta<\lambda.$

As $\lambda<\omega^*(R)$, the sequence (2) is not maximal. Therefore the point p is an interior point of $\bigcap_{\beta<\lambda} K[v_\beta(p_\beta)]$ with respect to T^*.

From the above lemma and from the fact that the kernels satisfy the axiom (C), we have the famous theorem of Baire modified as follows:

Theorem (Baire). *When we consider the topology T^*, any non empty open set*[6] *in the strongly complete ranged space must be of 2nd category.*[7]

Proof. To prove this theorem, it is sufficient to show that $\bigcap_\alpha A_{2\alpha}$, $0\leq\alpha<\omega_\nu^*\,(<\omega^*(R))$, where every $A_{2\alpha}\neq\phi$ is open and everywhere dense, is always everywhere dense.

Let G be an arbitrary non empty open set. We shall show that there exists a fundamental sequence of kernels

$$(1) \qquad K[v_0(p_0)]\supseteq\cdots\supseteq K[v_\alpha(p_\alpha)]\supseteq\cdots,\ \alpha<\omega_\nu^*,$$

satisfying following conditions:

(i) $\qquad K[v_\alpha(p_\alpha)]\subseteq(A_\alpha\cap G)$, for any even α,

(ii) $\qquad v_\alpha(p_\alpha)\in\mathfrak{V}_{\gamma_\alpha},\qquad \gamma_0<\gamma_1<\cdots<\gamma_\alpha<\cdots,$

(iii) $\qquad p_\alpha=p_{\alpha+1}.$

As A_0 is everywhere dense, $A_0\cap G$ is a non empty open set. Therefore, there exist a point $p_0\in(A_0\cap G)$ and $v_0(p_0)$ such as $K[v_0(p_0)]\subseteq(A_0\cap G)$.

Suppose that we have defined already the sequence of kernels $K[v_0(p_0)]\supseteq\cdots\supseteq K[v_\beta(p_\beta)]\supseteq\cdots$, $\beta<\alpha$, having the above properties.

In the case α is an even number (including limit number), by virtue of the lemma, we have that $\bigcap_{\beta<\alpha} K[v_\beta(p_\beta)]$ is open. As R is strongly complete, $G_\alpha=\bigcap_{\beta<\alpha} K[v_\beta(p_\beta)]$ is not empty, therefore $A_\alpha\cap G_\alpha$ is a non empty open set. Then, there exist a point $p_\alpha\in(A_\alpha\cap G_\alpha)$ and $v(p_\alpha)$ such as $K[v(p_\alpha)]\subseteq(A_\alpha\cap G_\alpha)$. And so, there exists $v_\alpha(p_\alpha)$ of

6) The open set with respect to T^* is not necessary to be open with respect to T. (On the other hand, it is clear that any open set with respect to T is also open with respect to T^*.)

7) The set M is called to be of 1st category with respect to T^*, if M can be expressed as follows: $M=\bigcup_\alpha M_\alpha$, $0\leq\alpha<\omega_\nu^*$ $(\leq\omega^*(R))$, where the sets M_α are non dense with respect to T^* (i.e. $\overline{R-\overline{M_\alpha}}=R$) (cf. 1)).

rank γ_α such as $v_\alpha(p_\alpha)\subseteq v(p_\alpha)$, for R is a ranged space. Then we have $K[v_\alpha(p_\alpha)]\subseteq K[v(p_\alpha)]\subseteq(A_\alpha \cap G_\alpha)\subseteq(A_\alpha \cap K[v_0(p_0)])\subseteq(A_\alpha \cap G)$.

If α is an odd number, put into $p_\alpha=p_{\alpha-1}$. As R is a ranged space and as $K[v_{\alpha-1}(p_{\alpha-1})]=K[_{\alpha-1}(p_\alpha)]$, there exists $v_\alpha(p_\alpha)$ of rank γ_α such as $K[v_\alpha(p_\alpha)]\subseteq K[v_{\alpha-1}(p_{\alpha-1})]$.

Thus, we obtained the desired fundamental sequence of kernels (1). Then, $\phi=\bigcap_{\alpha<\omega_\nu^*} K[v_\alpha(p_\alpha)]\subseteq \bigcap_{\alpha<\omega_\nu^*} K[v_{2\alpha}(p_{2\alpha})]\subseteq(\bigcap_{\alpha<\omega_\nu^*} A_{2\alpha}\cap G)$.

Example. Let $v_n(O)$, $n=1, 2, 3,\cdots$, be the sets of all points in Euclidean plane E^2 having coordinates (x, y) as follows:
$|xy|<1/n$, $|x|<a+1/n$, $|y|<b+1/n$, where a and b are two fixed positive numbers. Let $v_n(P)$, $P\in E^2$, be the sets of all points having coordinates as follows; $(x, y)=P+(\xi, \eta)$,[8] where $(\xi, \eta)\in v_n(O)$. Then the system $\{v\}$ defines the depth $\omega(E^2, P)=\omega$ at any point $P\in E^2$. In this case, the kernel of $K[v_n(O)]$ is the set of all points having coordinates (x, y) such as $|x|<1/1+na$, $|y|<1/1+nb$. It is clear that there is no $v_m(P)$ such as $v_m(P)\subseteq K[v_n(P)]$ for a large number n. Therefore the topology T^* is strictly stronger than the topology T, but in this case it is clear that $\omega^*(E^2, P)=\omega(E^2, P)=\omega$ for any point $P\in E^2$.

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8) $P+(\xi, \eta)$ means $(p+\xi, q+\eta)$ where $(p, q)=P$.