27. On the Property of Lebesgue in Uniform Spaces. VI

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Let S be a topological space. A covering of S is a family of open sets whose union is S. A covering is called *finite*, if it consists of a finite family.

Let us consider a *separated* uniform space S with a filter of surroundings $\mathfrak{S}$. A covering $\mathfrak{F}$ of S is said to have the *Lebesgue property* if there is a surrounding V in $\mathfrak{S}$ such that, for each point x of S, we can find an open set 0 of $\mathfrak{F}$ satisfying $V(x)\subset 0$.

We say that a separated uniform space has the *finite Lebesgue property* if any finite covering has the Lebesgue property. If any covering of S has the Lebesgue property, the space S is said to have the Lebesgue property. Such a space was studied by K. Iséki [4] and S. Kasahara [5]. S. Kasahara ([5], p. 129) has proved that *every uniform space having the Lebesgue property is complete.* On the other hand, the present author ([4], V, p. 619) has shown that the finite Lebesgue property does not imply the Lebesgue property and the existence of a non-complete uniform space having the finite Lebesgue property.

In this Note, we shall prove the following

Theorem 1. If the completion of a uniform space having finite Lebesgue property is normal, it has the finite Lebesgue property.

As easily seen, the converse of Theorem 1 is not true. There are non-normal complete uniform spaces (J. Dieudonné [2]).

To prove this suppose that $\hat{S}$ is the completion of a uniform space S having the finite Lebesgue property. According to a theorem of my Note ([4], IV, p. 524), it is sufficient to prove the following proposition.

Every bounded continuous function on $\hat{S}$ is uniformly continuous.

Let $f(x)$ be a continuous function on $\hat{S}$, then the restricted function $f(x|S)$ on S is uniformly continuous. Therefore, $f(x|S)$ is uniform continuously extended on $\hat{S}$ and it coincides with $f(x)$. Thus $f(x)$ is uniformly continuous, and $\hat{S}$ has the finite Lebesgue property.

Under the assumption of Theorem 1, we shall consider the relation between the dimension of S and its completion $\hat{S}$. There are some definitions of dimension for a topological space. However,

E. Hemmingsen [3] proved that some of these definitions are equivalent for normal space. Therefore, we shall use Lebesgue's covering definition by open sets. The *order* of a covering $\mathfrak{F}$ is the maximum number n such that there are $n+1$ sets of $\mathfrak{F}$ with non-empty intersection. A covering $\mathfrak{F}_2$ is called a *refinement* of a covering $\mathfrak{F}_1$ if to each member U of $\mathfrak{F}_2$, there is a member V of $\mathfrak{F}_1$ such that $U\subset V$. By the *dimension* (by Lebesgue) of a space S, we shall mean the minimum number n such that, for any finite covering $\mathfrak{F}$ of S, there is a finite covering $\mathfrak{F}'$ of order n and $\mathfrak{F}'$ is a refinement of $\mathfrak{F}$. By $\dim S$, we shall denote the dimension of S.

We turn now to prove the following

Theorem 2. If the completion $\hat{S}$ of a uniform space S having the finite Lebesgue property is normal, then $\dim S=\dim\hat{S}$.

To prove Theorem 2, we shall show

Lemma. Any uniform space S having the finite Lebesgue property is combinatorially imbedded in the completion $\hat{S}$ in the strong sense.

The notion stated in the conclusion is due to E. Čech and J. Novak [1].

Proof. By a theorem of my Note ([4], III), S is normal, therefore, regular. Let F_1, F_2 be closed sets in S. Then we prove $\overline{F_1\cap F_2}=\overline{F}_1\cap\overline{F}_2$, where the closure takes in $\hat{S}$, and this shows that S is combinatorially imbedded in $\hat{S}$ in the strong sense. It is clear that $\overline{F_1\cap F_2}\subset\overline{F}_1\cap\overline{F}_2$. Let $x\in\overline{F}_1\cap\overline{F}_2-\overline{F_1\cap F_2}$, then, by the regularity of S, we can take a neighbourhood G of x in $\hat{S}$ such that $\overline{G}\cap F_1\cap F_2=0$. Let $G_1=\overline{G}\cap F_1$, $G_2=\overline{G}\cap F_2$, then $x\in\overline{G}_1\cap\overline{G}_2$ and G_1 and G_2 are disjoint and closed in S. By the normality of S, there is a bounded continuous function f on S such that f is 0 on G_1, and f is 1 on G_2. By the assumption of S, f is uniformly continuous and therefore f is continuously extended on $\hat{S}$. This implies $\overline{G}_1\cap\overline{G}_2=0$, which contradicts $\overline{G}_1\cap\overline{G}_2\ni x$.

To prove Theorem 2, we shall prove the following theorem which is a generalisation of Theorem 2.

Theorem 3. Let S_1, S_2 be normal spaces. If $\overline{S}_1=S_2$ and S_1 is combinatorially imbedded in S_2 in the strong sense, then $\dim S_1=\dim S_2$.

A special case of Theorem 3 has been proved by M. Katětov [6]. We shall prove Theorem 3 by using a similar method.

Proof. Let dim $S_2 \leq n$, and $\mathfrak{F}=\{G_1, \cdots, G_n\}$ a finite covering of S_1. $\mathfrak{F}$ is shrinkable, by the normality of S_1. Let a covering $\mathfrak{F}'=\{H_1,\cdots, H_m\}$ of S_1 such that $\overline{H}_i \subset G_i\,(i=1, 2,\cdots, m)$, and $0_i = S_2 - \overline{S_1 - H_i}$, then, since S_1 is combinatorially imbedded in S_2 in the strong sense, $\bigcup_{i=1}^{m} 0_i = S_2$. Hence the covering $\{0_i\}$ has a refinement $\mathfrak{F}''=\{U_j\}$ of order $\leq n$. The covering $\{U_j \frown S_1\}$ of S_1 is a refinement of $\mathfrak{F}$ and order $\leq n$.

Next suppose dim $S_1 \leq n$. Let $\mathfrak{F}=\{G_1,\cdots, G_m\}$ be a covering of S_2. By the normality of S_2, $\mathfrak{F}$ is shrinkable, and let $\mathfrak{F}'=\{H_i\}$ be a covering of S_2 such that $\overline{H}_i \subset G_i\,(i=1, 2,\cdots, m)$. Then we can find a covering $\mathfrak{F}''=\{0_i\}$ of S_1 of order $\leq n$, and to each 0_i, there is an open set H_j such that $0_i \subset H_j$. Let $U_i = S_2 - \overline{S_1 - 0_i}$, then we have

$$\bigcup_{i=1}^{m} U_i = S_2 - \bigcap_{i=1}^{m} \overline{S_1 - 0_i} = S_2,$$

by our assumption. For some j, $\bigcap 0_j = 0$ implies $\bigcap U_j = 0$, and this shows that the order of the covering $\{0_i\}$ is not greater than n. If $0_i \subset H_j$, then $U_i \subset \overline{H}_j \subset G_j$, and the covering $\{U_i\}$ of S_2 is a refinement of $\mathfrak{F}$. This completes the proof.

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