18. *Note on Leontief's Dynamic Input-Output System*[1)]

By Hirofumi UZAWA

(Comm. by Z. SUETUNA, M.J.A., Feb. 13, 1956)

Leontief [2] raises the problem whether there exists a solution of the dynamic input-output system, when a flexible accelerator is introduced so that demand for capital equipment is proportional to the rate of change of the output when the latter is rising but zero when it is falling. In this paper, we shall concern ourselves with the condition, which must be imposed upon the input matrix and the stock matrix in order to exist a solution of the system.

1. Let there be a closed economy with n industries, $x_i(t)$ be the annual rate of total output of industry i, and $s_{ik}(t)$ be the stock of a commodity produced by industry i and used by industry k at time t. Then, if we suppose that all kinds of stocks are *irreversible*, *Leontief's dynamic input-output system* can be represented by the following $(*)$-*system* of differential equations:[2)]

$$(*)\qquad \begin{cases} x_i-\sum_{k=1}^{n}a_{ik}x_k=\sum_{k=1}^{n}\dot{s}_{ik}, & (i=1,\cdots,n),\\ \dot{s}_{ik}=\begin{cases} b_{ik}\dot{x}_k, & s_{ik}=b_{ik}x_k \text{ and } \dot{x}_k>0,\\ 0, & s_{ik}=b_{ik}x_k \text{ and } \dot{x}_k\leqq 0,\\ 0, & s_{ik}>b_{ik}x_k, \quad (i,k=1,\cdots,n), \end{cases} \end{cases}$$

where a_{ik} is the *input coefficient*, i.e. the amount of the product of industry i absorbed annually by industry k per unit of x_k, and b_{ik} is the *capital coefficient*, i.e. the stock of the product of industry i used per unit of the annual output of industry k. Upon these coefficients, we impose the following restrictions:

$$a_{ik}\geqq 0,\ b_{ik}\geqq 0, \quad (i,k=1,\cdots,n),$$
$$\sum_{k=1}^{n}a_{ik}\leqq 1, \quad (i=1,\cdots,n),$$
$$\sum_{k=1}^{n}a_{ik}<1 \text{ at least for one } i.$$

By using the matricial notation, we may write the first equation as follows:

$$(I-A)x=\dot{S}\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$$

1) Acknowledgement is due to Professor Kenneth J. Arrow, who suggested the author to use the method in [4] in handling the present problem.

2) $\dot{s}_{ik}=\frac{ds_{ik}}{dt}$ will stand for the rate of change of the variable s_{ik} with respect to time t.

where $x=[x_i]$ is the output vector, $S=[s_{ik}]$ the stock matrix and I, A are $n\times n$ matrices:

$$I=\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}, \qquad A=[a_{ik}].$$

By an initial position (x^0, S^0) of the $(*)$-system we shall mean only a pair of an n-vector $x^0=[x_i^0]$ and an $n\times n$ matrix $S^0=[s_{ik}^0]$ such that

$$x_i^0\geqq 0,\ s_{ik}^0\geqq 0 \text{ and } s_{ik}^0\geqq b_{ik}x_k^0,\ (i, k=1,\cdots, n).$$

A solution $(x(t), S(t))$ of the $(*)$-system with some initial position (x^0, S^0) is said to be *regular*, when the following condition is satisfied:

If $x_i(t_\nu)=0$ $(\nu=1, 2,\cdots)$ for some sequence $\{t_\nu\}$ such that $t_\nu>0$ and $\lim\limits_{\nu\to\infty} t=0$, then there exists a positive number $\bar{t}$ such that $\dot{x}_i(t)=0$ for $0<t<\bar{t}$.

2. Let I^0 be a set of pairs of indices (i, k) and II^0 be the set of pairs of indices which do not belong to I^0. Then, by the (I^0, II^0)-*system* will be meant the following system of differential equations for the unknowns $x_1,\cdots, x_n$:

$$x_i-\sum_{k=1}^{n} a_{ik}x_k=\sum_{k=1}^{n} b_{ik}^0 \max\ (\dot{x}_k, 0),\ (i=1,\cdots, n),$$

where

$$b_{ik}^0=\begin{cases} b_{ik}, & (i, k)\in I^0,\\ 0, & (i, k)\in II^0.\end{cases}$$

With the matricial notation, we have[3]

$$(I-A)\cdot x=B^0\cdot\max\ (\dot{x}, 0)$$

or

$$x=C\cdot\max\ (\dot{x}, 0),$$

where

$$B^0=[b_{ik}^0] \text{ and } C=(I-A)^{-1}B^0.$$

By the theorem of Frobenius,[4] $(I-A)$ is non-singular and the components of $(I-A)^{-1}$ are all non-negative, so that the components of C are non-negative.

For any initial position (x^0, S^0), let I^0 and II^0 be as follows:

$$I^0=\{(i, k);\quad s_{ik}^0=b_{ik}x_k^0\},$$
$$II^0=\{(i, k);\quad s_{ik}^0>b_{ik}x_k^0\}.$$

Then the (I^0, II^0)-system is said to be *associated with* the $(*)$-system at (x^0, S^0).

Concerning the solutions of the $(*)$-system and of the associated (I^0, II^0)-system, we have obviously the following

3) Max $(x, 0)$ will represent the vector $\xi=[\xi_i]$, where $\xi_i=\max\ (\dot{x}_i, 0)$.

4) Cf. Wielandt [5].

Theorem 1. *For any initial position (x^0, S^0), the $(*)$-system has a regular solution in the small neighborhood, if and only if the associated (I^0, II^0)-system has a regular solution in the small neighborhood of the initial position x^0. In this case, both solutions $x(t)$ do coincide.*

If there exists a solution of the (I^0, II^0)-system with an initial position x^0, then we must have

$$x^0 \in [C],$$

where $[C]$ is the set of all vectors x such that[5]

$$x = C\xi, \quad \xi \geqq 0.$$

Hence an initial position x^0 is said to be *admissible*, if x^0 belongs to the set $[C]$. If x^0 is admissible, we naturally have $x^0 \geqq 0$.

3. We shall consider the case, in which the matrix B^0 is non-singular. In this case, the (I^0, II^0)-system can be written as follows:

$$\max(x, 0) = \Gamma x, \quad \text{where } \Gamma = C^{-1}.$$

An initial position x^0 is admissible if and only if $\Gamma x^0 \geqq 0$.

A non-singular matrix C is said to be *retrograde*, if, for the inverse matrix $\Gamma = [\gamma_{ik}]$ of C, the inequation

$$\begin{bmatrix} \gamma_{i_1\nu} \\ \vdots \\ \gamma_{i_s\nu} \end{bmatrix} \geqq \begin{bmatrix} \gamma_{i_1i_1} & \cdots & \gamma_{i_1i_s} \\ \vdots & & \vdots \\ \gamma_{i_si_1} & \cdots & \gamma_{i_si_s} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_s \end{bmatrix}$$

has a non-negative solution $(\sigma_1, \cdots, \sigma_s)$ for any $s, i_1, \cdots, i_s$ and $\nu = j_1, \cdots, j_{n-s}$ such that

$$\{i_1, \cdots, i_s, j_1, \cdots, j_{n-s}\} = \{1, \cdots, n\}.$$

In the case of $n=2$, a matrix C with non-negative elements is retrograde, if and only if the determinant of C is negative: $\det C < 0$. Since $\det(I-A) > 0$, this is equivalent to $\det B^0 < 0$.[6]

In the case of $n=2$ or 3, a retrograde matrix with non-negative elements is negative-Hicksian.[7]

With respect to the solvability of the (I^0, II^0)-system, we can obtain the following

Theorem 2. *Let $C = (I-A)^{-1}B^0$ be non-singular. If there exists a solution of the (I^0, II^0)-system for any admissible initial position, then the matrix C is retrograde.*

Proof. Let $x(t)$ be a solution of the (I^0, II^0)-system with an initial position x^0 such that $\Gamma x^0 = \xi^0$, where

$$\xi^0_{i_\nu} = 0 \qquad (\nu = 1, \cdots, s),$$
$$\xi^0_{j_\nu} > 0 \qquad (\nu = 1, \cdots, n-s).$$

5) We write as usual $x \geqq y$ for vectors x and y, when $x_i \geqq y_i$ for $i = 1, \cdots, n$.
6) Cf. Georgescu-Roegen [1].
7) Cf. Samuelson [3].

Then

$$\dot{x}^0_{i_\nu} = -\sigma^0_{i_\nu} \leqq 0, \qquad (\nu=1,\cdots,s),$$
$$\dot{x}^0_{j_\nu} = \xi^0_{j_\nu} > 0, \qquad (\nu=1,\cdots,n-s)$$

and

$$\sum_{k=j_1,\cdots,j_{n-s}} \gamma_{i_\nu k}\xi^0_k - \sum_{k=i_1,\cdots,i_s} \gamma_{i_\nu k}\sigma^0_k \geqq 0.$$

Since this holds for any $\xi^0_{j_1},\cdots,\xi^0_{j_{n-s}}$ such that $\xi^0_{j_1}>0,\cdots,\xi^0_{j_{n-s}}>0$, there exists a vector $(\sigma_1,\cdots,\sigma_{n-s})$ so that

$$(\sigma_1,\cdots,\sigma_{n-s}) \geqq 0$$

and

$$\gamma_{i_\nu j} \geqq \sum_{k=1}^{n-s} \gamma_{i_\nu k}\sigma_k, \qquad \text{q.e.d.}$$

Theorem 3. *Let Γ be the inverse matrix of C. If*

$$\gamma_{ik} \leqq 0 \quad \textit{for } i \neq k;\ i,k=1,\cdots,n,$$
$$\gamma_{ii} > 0 \quad \textit{for } i=1,\cdots,n,$$

then the matrix C is retrograde and a regular solution of the (I^0, II^0)-system with any admissible position can be given by the solution of the following system:

$$\dot{x} = \Gamma x.$$

This is easily seen from the definition of the retrograde matrix.

References

[1] Georgescu-Roegen, N.,: Relaxation phenomena in linear dynamic models, Activity Analysis of Production and Allocation, ed. by T. C. Koopmans, New York (1951).
[2] Leontief, W. W.,: Studies in the Structure of the American Economy, Chap. 3, New York, 53–90 (1953).
[3] Samuelson, P. A.,: Foundations of Economic Analysis, Camb. Mass. (1948).
[4] Uzawa, H.,: Note on a gradient method (to be published).
[5] Wielandt, H.,: Unzerlegbare, nicht-negative Matrizen, Math. Zeit., **52**, 642–648 (1950).