37. *On Closed Mappings and Dimension*

By Kiiti MORITA
Tokyo University of Education, Tokyo

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1. Introduction. Let X be a normal space. We shall denote by "$\dim X$" the covering dimension of X and by "$\operatorname{ind}\dim X$" the inductive dimension of X which is defined by separation of closed sets; $\dim X \leqq n$ if every finite open covering of X has an open refinement of order $\leqq n+1$, and $\operatorname{ind}\dim X \leqq n$ if for any pair of a closed set F and an open set G with $F \subset G$ there exists an open set V such that $F \subset V \subset G$, $\operatorname{ind}\dim(\overline{V}-V) \leqq n-1$, where by definition $\operatorname{ind}\dim X = -1$ if and only if X is empty.

In this paper we shall establish the following generalizations of W. Hurewicz's theorems.[1]

Theorem 1. *Let f be a closed continuous mapping of a normal space X onto a normal space Y such that the inverse image $f^{-1}(y)$ consists of at most $k+1$ points for each point y of Y. Then we have*

$$\dim Y \leqq \operatorname{ind}\dim X + k.$$

Theorem 2. *Let f be a closed continuous mapping of a normal space X onto a paracompact T_1-space Y such that*

$$\dim f^{-1}(y) \leqq m$$

for each point y of Y. Then

$$\dim X \leqq \operatorname{ind}\dim Y + m.$$

2. Lemmas. Let $\mathfrak{G}$ be an open covering of a space X and A a subset of X. We shall write $(\mathfrak{G})\text{-}\dim A \leqq n$ if there exists an open covering of a subspace A which has an order $\leqq n+1$ and is a refinement of $\mathfrak{G}$.

Lemma 1. *Let X be a normal space. Then we have* $\dim X \leqq n$ *if and only if, for any pair of a closed set F and an open set G with $F \subset G$ and for any finite open covering $\mathfrak{G}$ of X, there exists an open set V such that*

$$F \subset V \subset G, \quad (\mathfrak{G})\text{-}\dim(\overline{V}-V) \leqq n-1.$$

This is proved in [4]. From this lemma we get immediately Lemma 2 which is due to N. Vedenisoff.

Lemma 2. *If X is a normal space, then we have*

$$\dim X \leqq \operatorname{ind}\dim X.$$

In case A is a closed subset of a normal space X, we shall

1) W. Hurewicz proved these theorems for the case where X and Y are separable metric spaces. Cf. [2], [3]. In [7] we have used Theorem 1 for the case of metric spaces.

write $\dim(X, A) \leqq n$ if $\dim F \leqq n$ for every closed set F of X such that $F \subset X - A$. From the proof of [4, Theorem 2.2] we obtain Lemma 3 below, and Lemma 4 is a direct consequence of Lemma 3 and the sum theorem.

Lemma 3. *Let A be a closed set of a normal space X and $\mathfrak{G}$ a finite open covering of X. If*

$$(\mathfrak{G})\text{-}\dim A \leqq n, \quad \dim(X, A) \leqq n,$$

then

$$(\mathfrak{G})\text{-}\dim X \leqq n.$$

Lemma 4. *If A is a closed set of a normal space X, then* $\dim X = \mathrm{Max}(\dim A, \dim(X, A))$. *More generally, if $\{A_i\}$ is a countable closed covering of X such that $A_i \subset A_{i+1}$, $i = 1, 2, \cdots$, then* $\dim X = \mathrm{Max}(\dim(A_i, A_{i-1}))$ *where we put $A_0 = 0$.*

Lemma 5. *Let X be a normal space and $\mathfrak{G}$ a locally finite open covering of X. Then we have $(\mathfrak{G})$-$\dim X \leqq n$ if and only if there exist $n+1$ closed (or open) subsets P_i, $i = 0, 1, \cdots, n$, such that*

$$X = \bigcup_{i=0}^{n} P_i, \quad (\mathfrak{G})\text{-}\dim P_i \leqq 0, \ i = 0, 1, \cdots, n.$$

Proof (cf. [4]). Let $(\mathfrak{G})$-$\dim X \leqq n$ and $\mathfrak{G} = \{G_\alpha \mid \alpha \in \Omega\}$. Then there exists an open covering $\{U_\alpha\}$ of X with order $\leqq n+1$ such that $U_\alpha \subset G_\alpha$ for each α. We take further an open covering $\{V_\alpha\}$ of X such that $\overline{V}_\alpha \subset U_\alpha$ for each α. If we put

$$P_0 = \bigcup \Big(\bigcap_{i=0}^{n} \overline{V}_{\alpha_i}\Big), \quad Q_0 = \bigcup \Big(\bigcap_{i=0}^{n} V_{\alpha_i}\Big),$$

where the sum is taken over all systems of $n+1$ distinct indices $\alpha_0, \alpha_1, \cdots, \alpha_n$ from Ω, then P_0 is closed and

$$(\mathfrak{G})\text{-}\dim P_0 \leqq 0, \quad (\mathfrak{G})\text{-}\dim(X - Q_0) \leqq n-1,$$

since the order of $\{\bigcap_{i=0}^{n} U_{\alpha_i} \mid \alpha_i \in \Omega, \ i = 0, \cdots, n\} \leqq 1$ and the order of $\{(X - Q_0) \cap V_\alpha \mid \alpha \in \Omega\} \leqq n$. By repeated application of this process we have a decomposition desired in the lemma. It is obvious that for each i there exists an open set P_i^* such that $P_i \subset P_i^*$, $(\mathfrak{G})$-$\dim P_i^* \leqq 0$.

Conversely, if there is such a decomposition, we have clearly $(\mathfrak{G})$-$\dim X \leqq n$.

3. Proof of Theorem 1. We shall prove Theorem 1 by induction on $\operatorname{ind}\dim X = n$. The theorem is trivially true in case $\operatorname{ind}\dim X = -1$. We shall assume the theorem for $\operatorname{ind}\dim X \leqq n-1$.

Let $\operatorname{ind}\dim X = n$. If $k = 0$, we see by Lemma 2 that the theorem holds. We shall prove the theorem for $k = k_0$ assuming it for $k \leqq k_0 - 1$.

For any pair of a closed set F and an open set G of Y with $F \subset G$ we shall prove the existence of an open set V of Y such that

$$(1) \qquad F \subset V \subset G, \quad \dim(\bar{V}-V) \leqq n+k_0-1.$$

By the assumption that $\operatorname{ind}\dim X=n$, there exists an open set H of X such that $f^{-1}(F) \subset H \subset f^{-1}(G)$, $\operatorname{ind}\dim(\bar{H}-H) \leqq n-1$. Let us put $V=Y-f(X-H)$. Then we have

$$(2) \qquad \bar{V}-V \subset f(\bar{H})-V, \quad F \subset V \subset G.$$

If we put $K=f(\bar{H})-V$, $K_1=f(\bar{H}-H)-V$, then by the assumption of induction (concerning $\operatorname{ind}\dim X$) we have $\dim K_1 \leqq n-1+k_0$, since $\operatorname{ind}\dim(\bar{H}-H) \leqq n-1$ and the partial mapping $f\,|\,(\bar{H}-H) \frown f^{-1}(K_1)$ is closed.

Let M be any closed set of K (and hence of Y) contained in $K-K_1$. If we denote by f_1 the partial mapping of f whose domain is $(X-H) \frown f^{-1}(M)$ and whose range is M, then f_1 is a closed onto mapping such that $f_1^{-1}(y)$ consists of at most k_0 points for each point y of M, since $M \subset K-K_1 \subset f(H)-V \subset f(H) \frown f(X-H)$. Hence by the assumption of induction on k we have $\dim M \leqq n+k_0-1$, since $\operatorname{ind}\dim(X-H) \frown f^{-1}(M) \leqq \operatorname{ind}\dim X \leqq n$. Therefore $\dim(K, K_1) \leqq n+k_0-1$.

We now apply Lemma 4 to our case and we get $\dim K \leqq n+k_0-1$ and hence

$$(3) \qquad \dim(\bar{V}-V) \leqq n+k_0-1.$$

By (2) and (3) we see that V satisfies the condition (1). By Lemma 1 we have $\dim X \leqq n+k_0$. This completes our proof.

4. Theorem 3. *Under the same assumption as in Theorem 1, if* $\dim X \leqq 1$, *we have* $\dim Y \leqq \dim X+k$.

Proof. In case $k=0$ the theorem holds clearly. Assume that the theorem holds for $k<k_0$; we shall prove the theorem for $k=k_0$. Let F and G be a closed and an open sets of Y such that $F \subset G$ and let $\mathfrak{G}$ be any finite open covering of Y. We put $\mathfrak{H}=\{f^{-1}(U)\,|\,U \in \mathfrak{G}\}$. Let $\dim X=1$. By Lemma 1 there exists an open set H of X such that $f^{-1}(F) \subset H \subset f^{-1}(G)$, $(\mathfrak{H})\text{-}\dim(\bar{H}-H) \leqq 0$. If we put $V=Y-f(X-H)$, $K=f(\bar{H})-V$, $K_1=f(\bar{H}-H)-V$, we have $F \subset V \subset G$, $(\mathfrak{G})\text{-}\dim K_1 \leqq k_0$, while $\dim(K, K_1) \leqq k_0$ by the assumption of induction. Thus we have $(\mathfrak{G})\text{-}\dim(\bar{V}-V) \leqq k_0$ by Lemma 3; this shows by Lemma 1 that $\dim Y \leqq k_0+1$.

Remark. In case X is a totally normal space in the sense of C. H. Dowker [1] it can be shown that under the same assumptions as in Theorem 1 we have $\operatorname{ind}\dim Y \leqq \operatorname{ind}\dim X+k$.

5. Proof of Theorem 2. We shall carry out our proof by induction on $\operatorname{ind}\dim Y$. The theorem is trivially true if $\operatorname{ind}\dim Y=-1$. Assume the theorem for $\operatorname{ind}\dim Y \leqq n-1$. Let $\operatorname{ind}\dim Y=n$.

Let $\mathfrak{G}$ be any finite open covering of X. By the assumption of the theorem, for each point y of Y there exists an open set H_y of X such that

(4) $$(\mathfrak{G})\text{-dim}\, H_y \leqq m, \quad f^{-1}(y) \subset H_y.$$

If we put $V_y = Y - f(X - H_y)$, then V_y is an open neighbourhood of y and

(5) $$f^{-1}(y) \subset f^{-1}(V_y) \subset H_y.$$

Since Y is paracompact, there exists a locally finite open covering $\mathfrak{U} = \{U_\alpha \mid \alpha \in \Omega\}$ which is a refinement of $\{V_y \mid y \in Y\}$. The space Y is normal as the image of a normal space under a closed continuous mapping. Hence there is a closed covering $\{F_\alpha \mid \alpha \in \Omega\}$ of Y such that $F_\alpha \subset U_\alpha$ for each α.

Since $\text{ind dim}\, Y = n$, there exists for each α an open set W_α such that $F_\alpha \subset W_\alpha$, $\overline{W}_\alpha \subset U_\alpha$, $\text{ind dim}\,(\overline{W}_\alpha - W_\alpha) \leqq n-1$.

Assuming that the set Ω of indices consists of all ordinals less than a fixed ordinal α_0, we put

$$H_1 = W_1;\ H_\alpha = W_\alpha - \bigcup_{\beta<\alpha} \overline{W}_\beta,\ \alpha > 1.$$

Then we have

(6) $$Y = \bigcup \{\overline{H}_\alpha \mid \alpha \in \Omega\}$$

and $\text{ind dim}\, \overline{H}_\alpha \cap \overline{H}_\beta \leqq n-1$ for $\alpha \neq \beta$, since $\overline{H}_\alpha \cap \overline{H}_\beta \subset \overline{W}_\beta - W_\beta$ if $\beta < \alpha$.

By the assumption of induction we have

(7) $$\dim f^{-1}(\overline{H}_\alpha) \cap f^{-1}(\overline{H}_\beta) \leqq m+n-1, \text{ for } \alpha \neq \beta.$$

On the other hand, for each α $\overline{H}_\alpha \subset U_\alpha$ and each U_α is contained in some V_y. Therefore we obtain by (4) and (5)

(8) $$(\mathfrak{G})\text{-dim}\, f^{-1}(\overline{H}_\alpha) \leqq m \leqq m+n.$$

Since $f^{-1}(\overline{H}_\alpha) \subset f^{-1}(U_\alpha)$ and $\{f^{-1}(U_\alpha)\}$ is a locally finite open covering of X, by [5, Theorem 3] we conclude from (7) and (8) that $(\mathfrak{G})\text{-dim}\, X \leqq m+n$. Therefore we have $\dim X \leqq m+n$ since $\mathfrak{G}$ is arbitrary, and hence the theorem holds for any Y with $\text{ind dim}\, Y = n$. This completes the proof.[2)]

6. Theorem 4. *Let f be a closed continuous mapping of a normal space X onto a paracompact T_1-space Y such that* $\dim f^{-1}(y) \leqq 0$ *for each point y of Y. Then* $\dim X \leqq \dim Y$.

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2) For the special case where X is an S_σ-space (any CW-complex is an S_σ-space; for the definition, cf. [6]) we can prove the relation $\dim X \leqq \text{ind dim}^*\, Y + m$ under the same assumption as in Theorem 2, where $\text{ind dim}^*\, Y$ means the inductive dimension of Y in the sense of Menger-Urysohn; this relation is proved also by K. Nagami independently.

Added in proof: He also proved Theorem 2 under a more restrictive assumption; cf. his forthcoming paper.

Proof. Let $\mathfrak{G}$ be any finite open covering of X. Then for each point y of Y there exists an open neighbourhood V_y of y such that

$$(\mathfrak{G})\text{-dim}\, f^{-1}(V_y) \leqq 0; \tag{9}$$

this is seen as in the proof of Theorem 2 (cf. (5)).

Let $\mathfrak{U}$ be a locally finite open covering of Y which is a refinement of $\{V_y \mid y \in Y\}$. Let $\dim Y = n$. Then by Lemma 5 there exist $n+1$ closed sets Q_i, $i=0,1,\cdots,n$ such that

$$Y = \bigcup_{i=0}^{n} Q_i;\ (\mathfrak{U})\text{-dim}\, Q_i \leqq 0,\ i=0,1,\cdots,n.$$

Since each set belonging to $\mathfrak{U}$ is contained in some V_y, it follows from (9) that $(\mathfrak{G})\text{-dim}\, f^{-1}(Q_i) \leqq 0$, $i=0,1,\cdots,n$. According to Lemma 5 this shows that $(\mathfrak{G})\text{-dim}\, X \leqq n$. Thus we have $\dim X \leqq n$.

From the above proof we obtain immediately

Lemma 6. *Let f be a continuous mapping of a normal space X onto a paracompact normal T_1-space Y and $\mathfrak{G}$ a locally finite open covering of X. If for every point y of Y there exists a neighbourhood $V(y)$ of y such that* $(\mathfrak{G})\text{-dim}\, f^{-1}(V(y)) \leqq 0$, *then* $(\mathfrak{G})\text{-dim}\, X \leqq \dim Y$.

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